THE GYROSCOPE

CORDEIRO

$$C\omega\dot{\theta} = A\ddot{\psi}$$

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BY

F. J. B. CORDEIRO

AUTHOR OF

"THE ATMOSPHERE," "BAROMETRICAL HEIGHTS," ETC.

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PREFACE.

Our exact knowledge of Rotary Motion, as of Dynamics in general, dates from the time of Newton. Euler, Laplace, Lagrange, Poisson and Poinsot are illustrious names in the development of the theory. Foucault, in 1855, demonstrated the rotation of the earth by means of the gyroscope, and gave it its name. Its practical applications date from yesterday. These began with the Griffin Grinding Mill, and have been followed by the Howell and Obry devices for keeping a torpedo on a straight course, the Schlick Stabilsator for ships, the Brennan Gyro-Monorail, the Anschuetz-Kaempfe Gyro-compass, and the end is not yet.

The theory of rotary motion is not simple, nor is it yet complete. Not all inventors have understood the reason for their devices, and not all mathematicians have had a clear conception of the theory, as evidenced by the unnecessary complication of their treatments.

Attempts have been made to explain gyroscopic action without mathematics, or at least without the Calculus. It is hardly necessary to say that all such attempts are futile. It is impossible to explain the actions of a gyroscope without mathematics, and it is impossible to understand them without such knowledge.

Many students are afraid of what is called the higher mathematics, and are permitted to avoid them in our higher institutions of learning. Mathematics, in its broadest sense, is the science of time, space, mass and force, and the

relations existing between these four quantities. It is the foundation upon which all the exact sciences are built, as it is the foundation of the universe. Everything else may and does change, but the principles of mathematics alone remain eternal.

There is no doubt that mathematics are difficult: all other forms of intellectual effort are mere child's play in comparison. Hence many who are scientifically inclined, seek a field in the inexact sciences, or in the pseudo-sciences, where these difficulties may be shirked. It is noteworthy, however, that even here, as these branches become developed, they are found to reach down to the solid bedrock of mathematics, where their cultivators, who have neglected the fundamental science of all, find themselves in an unenviable position. This is notably the case with meteorology.

The day will undoubtedly come (and the sooner, the better) when mathematics will be made the foundation of every education, and no man (or woman) can be considered educated who does not know the Calculus. A virile mind will not quail before its difficulties, but will experience a joy in surmounting its obstacles — the gaudium certaminis — such as can be found in no other intellectual field.

The student with an elementary knowledge of mathematics, who attempts to understand gyroscopics from a study of its scattered parts in standard treatises, and from the few monographs as yet written, will find the task tedious — probably repulsive. For this reason, it has seemed advisable to the author to write a monograph which may be easily understood by anybody possessing an elementary knowledge of mechanics and the calculus. The book is divided into two parts — the development of the theory from the *Fundamental Gyroscopic Principle*, and a discussion of its modern practical applications. The

motions of the heavenly bodies, where gyroscopics are exhibited in their grandest and freest (frictionless) form, have been fully explained, but the engineer and cursory student, who care only for the elementary theory and an explanation of its applications, may omit the astronomical discussion without loss of continuity.

F. J. B. C.

NEWTON CENTRE, MASS.,
June 16, 1913.

PART I.

THEORY.

THE GYROSCOPE

PART I.

THEORY.

1. The Name.

WE shall define any rotating mass as a gyroscope. Foucault gave this name to the instrument, consisting of a rapidly rotating flywheel, by which he demonstrated the rotation of the earth. We shall extend this name to any rotating mass. It is eminently suitable. The word gyroscope means simply any body exhibiting, or showing (σκωπειν), gyration, or rotation.

The term gyrostat, often used for gyroscope, is particularly objectionable. There is no such thing as a gyrostat, or instrument which maintains its plane of rotation. We shall see that no gyroscope can maintain its plane when acted upon by an outside force, although it is true that, provided it possesses a high rotational moment, it changes its plane much more slowly than a non-rotating body under similar conditions, thus exhibiting a high degree of what we might call rotational inertia. Here again it changes its plane in a totally different manner from a non-rotating body, but the slightest couple, acting for a sufficiently long time, will change its plane of rotation to any desired extent.

Certain German writers employ the word "top" (Kreisel), as a generic name for the gyroscope, although, happily, an increasing number are now using the word gyroscope. This seems to be an overstraining towards simplicity, such

as leads a man to call his palace a cottage. But a cyclone is a gyroscope, i.e., a rotating mass, and so is a child's hoop, and so is the earth; but none of these are tops, although it is true that a top is a gyroscope. Furthermore a top is a top only while a child is playing with it. Immediately a philosopher begins to experiment with it, it ceases to be a top and becomes a gyroscope.

The interest of gyroscopes for us, apart from certain recent applications, is manifold. We live in a universe of gyroscopes; everywhere there is rotary motion. We live *on* a gyroscope, and we ride on gyroscopes, be it carriage, bicycle, train or aeroplane. Our factories are full of gyroscopes, where occasionally ignorance of gyroscopic laws leads to disaster. The axle of a rapidly-rotating grindstone works loose and the stone flies apart, killing a workman. It is ascribed to simple centrifugal force, but the stone would have withstood this stress if preserving the same plane. It could not, however, withstand the enormous gyroscopic couple, acting at right angles to its plane when the axle moved.

An aviator makes too sharp a turn, thereby setting up a rotation of the whole machine about an axis which, in general, is not a principal axis. It becomes a gyroscope, even if the other gyroscopes he is carrying, viz., the rotary motor and propeller, are not turning. So that, even with the motor dead, and a fortiori with the motor turning, gyroscopic couples are set up in a direction which no human instinct can foresee, and he is capsized. It is ascribed to a "hole in the air," or some other "*theory*."

2. Mathematical Definitions.

If a body is rotating about some axis, the integral, $\int p^2\, dm$, where dm is an infinitesimal component of its

mass, and p its distance from the axis, is called its moment of inertia about that axis. If $\int p^2\, dm = Mk^2$, where M is the total mass of the body, then k is called the radius of gyration about that axis, and the equation means that, if the total mass were concentrated into a mathematical point at the extremity of the radius of gyration, the moment of inertia would be the same.

Through any point in a mass, three mutually perpendicular axes can be found, which are called the principal axes of inertia at that point. These axes possess the property that the moment of inertia about them is greater or less than that about any other axis in their immediate vicinity, and one of these principal axes is a maximum for the body, while another is a minimum. The principal axes at the center of inertia (or center of gravity) are called the *Principal Axes* of the body. In general, for an unsymmetrical body, the moments of inertia about these axes are unequal, and we shall define a body with three unequal principal axes as a *Tri-axial* body. If the body is symmetrically homogeneous about one axis, as in a homogeneous solid of revolution, two of these axes become equal, and we shall define such a body as a *Bi-axial* body. If the body is homogeneously symmetrical about all axes, as in a homogeneous sphere, we shall define it as *Uni-axial.**

A couple is defined as two equal forces acting in opposite directions, and at equal distances, about some axis. The moment of a force about an axis is defined as the product of the force into its distance from the axis. The moment of a couple or, as it is usually called, simply the *Couple*, is the product of one of the forces into the distance between them.

* In works on mechanics and optics, what we have defined as a bi-axial body, is called uni-axial. The above definition is more consistent.

The plane of the forces is the plane of the couple, and the axis about which they act, which is perpendicular to the plane of the couple, is called the axis of the couple. The word "*Torque*" is engineering "*slang*" for couple. It should never be used. A single name or symbol should be strictly preserved for every mathematical entity. It would be as logical for engineers to use a separate and different symbol for π, or for a man to be known to one set of acquaintances as Smith, and to another set as Brown.

A couple acting about an axis is subject to the same laws as a simple force acting along a straight line. That is, the couple is equal and opposite (Newton's third law) to the moment of inertia of the body about its axis into its angular acceleration about that axis, just as a simple force is equal and opposite to the mass of a body into its linear acceleration. The moment of momentum of a body about an axis is defined as its moment of inertia into the angular velocity about the axis, and this is evidently the *time* integral of the couple acting about the axis. The kinetic energy about any axis is defined as half the moment of inertia, into the square of the angular velocity, and is evidently the *angle* or *space* integral of the couple about that axis; or it is the equivalent of the work done by the couple.

It is evident that all these quantities, depending upon an axis having a certain definite direction, are *directed* quantities and, therefore, by the parallelogram principle, we can resolve them into components having other directions about other axes. Thus, when a body is turning about some axis, it can be considered to have a component of this turn about any other axis, and the amount of this component is found by multiplying the original turn into the cosine of the angle between the two axes.

There are two directions in which a body may turn about an axis — right and left rotation. If we adopt one

direction as positive, then the other is mathematically negative. The symbol plus applies to one, the symbol minus to the other.

It is important that what is regarded as *positive*, *standard* or *normal* rotation should be fixed by convention. The lack of such a convention has resulted in much confusion in mathematical figures and demonstrations. A priori there appears to be no reason for choosing one direction over the other. It might be argued for right rotation that most people are right-handed, that clocks turn to the right, and that most civilized nations make screws with a right twist.* All this is probably accidental. The more convincing argument is that in the Northern Hemisphere, where there is the most land, and the greatest population and the highest civilization exist, practically all the motions of nature are to the left — cyclones, heavenly bodies, etc. We shall therefore define as positive rotation that to the left, or against a clock.

It would be a great advantage if a uniform notation could prevail everywhere regarding gyroscopic motion. The following notation will be used throughout this book. A, B, C, represent the principal moments of inertia in ascending order. Velocities and accelerations, or first and second derivatives with respect to the time, will be represented by super-dots, and double super-dots, respectively. Thus, $\dot{\theta}$ represents the angular velocity of the angle θ, and $\ddot{\theta}$ its angular acceleration. The signs, $\perp$ and $\parallel$ will be used to designate perpendicularity and parallelism, respectively. Motion about an axis through the center of gravity of the body will be called rotation, while the motion of the center of gravity about an external axis will be called revolution. The symbol ω will be used generally to designate the

* This is not universal. Germans and Americans rifle their guns to the right; English and Italians, to the left.

angular velocity about the principal axis C, or the axis of greatest moment of inertia. In the case of a tri-axial body, ω_1, ω_2, ω_3 will be used to designate the angular velocities about the axes A, B, C, respectively.

3. Gyroscopic Action.

In Fig. 1 let us suppose a circular disc with its center of gravity O fixed, and capable of turning about this center in any direction. Let us suppose it is rotating about the

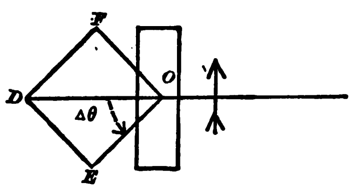

FIG. 1.

axis OD with angular velocity ω, in a positive direction. In other words, it is rotating about the axis of the disc, an axis $\perp$ to its plane through the center of gravity, in a direction to the left when viewed from D, as indicated by the arrow. The moment of inertia about this axis is a maximum for the disc, and will be designated by C. Let the line OD represent the moment of momentum about this axis, both in direction and magnitude. By the parallelogram principle we can resolve this moment of momentum OD into two component moments OE and OF, the line OE representing a moment of momentum, both in magnitude and direction about an axis making an angle $\Delta\theta$ with OD, and the line OF representing a simultaneous moment about a $\perp$ axis both in magnitude and direction.

Let us now start to turn the rotating disc about an axis through O, $\perp$ to the page, in a positive direction, i.e., in the direction indicated by the arrow. If the disc were

not rotating we should have no difficulty in turning the axis OD into coincidence with OE, but with the disc rotating it is a different matter. We do not yet know what would happen. Let us examine what occurs at the very first instant of turning. We can consider the angle $\Delta\theta$ to become as small as we please, and write it $d\theta$. Let us suppose that in the infinitesimally small interval of time dt, in which we turn the axis OD through the infinitesimal angle $d\theta$, this axis will remain in the plane of the page, and that the component moments of momentum OE and OF will not have had time to change. In other words, the moment of momentum about the direction OE will remain what it was, although the axis of the disc has changed from the direction OD into the direction OE.

Now such a supposition cannot be strictly true, although it will approach more and more to the truth as dt and $d\theta$ become smaller, and at the limiting value of zero for these quantities, it will be strictly true.

The rate at which the moment of momentum about an axis changes, measures the couple acting about that axis (Art. 2). The couple about the axis of the disc will therefore be $\dfrac{OD \cos d\theta - OD}{dt}$, and the couple about a $\perp$ axis, in the plane of the page, is $\dfrac{OD \sin d\theta - o}{dt}$. At the limit the first couple becomes zero, and the second couple becomes $OD\dfrac{d\theta}{dt} = C\omega\dot{\theta}$.

We find therefore that, if we turn the axis of a rotating body about a $\perp$ axis, the moment of momentum about the original axis will be unaffected. Further, although the body will start to turn in obedience to the turning couple, there will, however, immediately be set up a couple acting about an axis $\perp$ to the axis of the turning couple

and the original axis of rotation. This couple is called the gyroscopic couple, and its amount is $C\omega\dot{\theta}$, or it is equal to the original moment of momentum into the angular velocity of the ⊥ turn. The axis of this couple will be in the direction OF, i.e., it tends to turn the body positively about this axis, or to raise the axis of the disc into coincidence with the axis of turning. And if we continue turning about a fixed axis, the original axis of rotation will, in fact, be brought into coincidence with the turning axis. By the same reasoning, if we turn negatively, $\dot{\theta}$ is negative, and the gyroscopic couple $C\omega\dot{\theta}$ becomes negative, and the axis of the disc will be brought *down* ⊥ to the page into coincidence with the negative axis of turning.

This is the *Fundamental Gyroscopic Principle*, which may be stated thus: If we turn a rotation axis about a ⊥ axis, a couple will be set up tending to bring the axes into coincidence. If ω is the angular velocity about the original axis of rotation, $\dot{\theta}$ that about the ⊥ turning axis and $\ddot{\psi}$ is the angular acceleration about the gyroscopic axis, which is ⊥ to the other two, then the amount of the gyroscopic couple is $A\ddot{\psi} = C\omega\dot{\theta}$, where A is the moment of inertia about the gyroscopic axis. This follows from Newton's third law that action is equal to reaction. Due regard must be had to signs. If ω and $\dot{\theta}$ are not of the same sign, the couple will be negative. $C\omega\dot{\theta} = A\ddot{\psi}$ is the *Fundamental Gyroscopic Equation*, from which we can derive all the properties and motions of a gyroscope.

To recapitulate: If we turn a rotating body about an axis ⊥ to its rotation axis, a couple will be immediately set up having an axis ⊥ to the two former axes, and the amount of this couple will be the product of the rotational moment by the angular velocity of the turn, its sense being such as to tend to bring the rotational and turning axes into coincidence, both in sense and direction.

4. Gyroscopic Action the Result of Centrifugal Forces.

The gyroscopic couple we have just investigated is essentially a centrifugal force, or rather a compound centrifugal force. We are familiar with the centrifugal force due to a simple rotation which acts in the plane of rotation and always away from the axis of rotation. Its amount is $Mr\dot{\psi}^2$, where M is the mass being carried about the axis (supposed to be concentrated into a point), r its distance from the axis and $\dot{\psi}$ the angular velocity. We have now to deal with a compound centrifugal force due to the compounding of two simultaneous rotations. We shall see that this compound centrifugal force does not act in the plane of either of the rotations, but in a plane ⊥ to both.

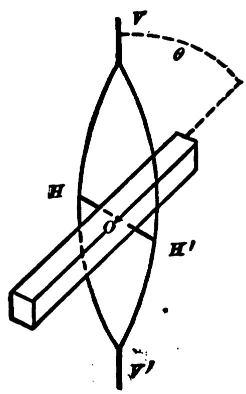

FIG. 2.

In Fig. 2, a bi-axial body, here a bar, is pivoted about a horizontal axis HH', through its center of inertia. This axis is held in a vertical frame which can turn about a vertical axis VV'. We shall suppose throughout this book that the supporting mechanism is without weight, and that there is no friction, unless otherwise expressly stated. In practice we can approximate these conditions to any desired extent. Let the inclination of the axis of the bar to the vertical be θ, A the moment of inertia about this axis and C that about a ⊥ axis. We now turn the mechanism about the vertical axis with an angular velocity $\dot{\psi}$. We can resolve this velocity into a component $\dot{\psi}\cos\theta$ about the axis of the bar, and a component $\dot{\psi}\sin\theta$ about a ⊥

axis. It is evident that gyroscopic couples will be set up since the body has rotations about two $\perp$ axes. One gyroscopic couple, $A\dot{\psi}\cos\theta, \cdot \dot{\psi}\sin\theta$, will tend to bring the axis into coincidence with the axis of $\dot{\psi}$; the other gyroscopic couple will act in the opposite direction, and is equal to $C\dot{\psi}\sin\theta \cdot \dot{\psi}\cos\theta$. The resulting couple is $(C - A)\dot{\psi}^2 \sin\theta\cos\theta$. But this is the well-known expression for the centrifugal couple acting in a vertical plane through the axis of the bar. The bar will therefore oscillate above and below the horizontal plane by an amount equal to the original inclination. By writing the result down at once from gyroscopic principles, we have saved ourselves a tedious integration.

As another example, let us suppose a sphere, Fig. 3, rotating about an axis OC with angular velocity ω, this

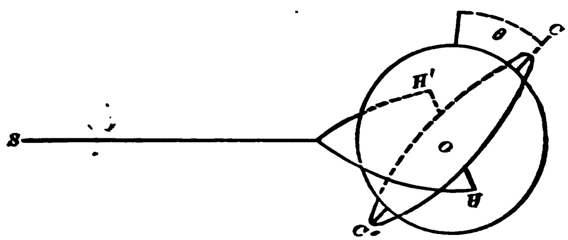

FIG. 3.

axis being held in a ring CC', which is pivoted about a horizontal axis HH' held in a fork SHH'. The ring can turn about the horizontal axis, and the whole is set revolving about the fixed point S, in a horizontal plane, with angular velocity $\dot{\psi}$. Let R be the radius of the sphere, D the distance SO, θ the inclination of the rotation axis to the vertical and the rotation and revolution in the same sense. We shall calculate the centrifugal forces and show that their sum is simply the gyroscopic couple we have previously obtained.

We can resolve the rotation ω into $\omega \cos\theta$ about a vertical axis, and $\omega \sin\theta$ about SO. Let us first consider a ring of radius r rotating about SO with angular velocity $\omega \sin\theta$. The velocity of any particle in the ring is $D\dot{\psi} - r\omega \sin\theta \cos\phi$, where ϕ is the angle any radius makes with the vertical. The centrifugal force outward from S will be $\frac{dm}{D}[D\dot{\psi} - r\omega \sin\theta \cos\phi]^2$, where dm is an elementary mass and equals $rd\phi \cdot \delta$, δ being the density of the substance. It is evident that there will be a couple about HH' since the velocity, and consequently the centrifugal force, in the lower half of the ring is greater than in the upper half.

The centrifugal moment of a particle about the axis HH' is $\frac{rd\phi \cdot \delta}{D}[D\dot{\psi} - r\omega \sin\theta \cos\phi]^2 r \cos\phi$.

The sum of these moments is

$$\int_0^{2\pi} \frac{r\delta}{D}[D\dot{\psi} - r\omega \sin\theta \cos\phi]^2 r \cos\phi \, d\phi = 2\pi r\delta \cdot r^2\omega \sin\theta\dot{\psi}.$$

But $2\pi r\delta$ is the mass of the ring $= m$, and $mr^2\omega \sin\theta$ is the moment of momentum of the ring about the axis SO. Hence the sum of the moments of the centrifugal forces about HH' is the moment of momentum about SO into the angular velocity, $\dot{\psi}$. This is at once seen to be nothing else than the gyroscopic couple tending to bring the rotational axis SO of the ring into coincidence with the axis of $\dot{\psi}$.

For a disc the centrifugal resultant would be

$$\sum_0^R 2\pi r^3\delta \cdot \omega \sin\theta\dot{\psi} \, dr = \frac{\pi R^4}{2}\delta \cdot \omega \sin\theta\dot{\psi}.$$

But $\pi R^2\delta$ is the mass of the disc and $\frac{R^2}{2}$ is k^2, where k is the radius of gyration. Hence the moment of momentum of

the disc about SO into the angular velocity $\dot{\psi}$ will still be the resultant of the centrifugal moments and this is what we know to be the gyroscopic couple.

For the whole sphere the centrifugal resultant is

$$\sum_{0}^{R} \pi r^4 \delta \cdot \omega \sin \theta \dot{\psi} \, dh,$$

where h is the distance of any disc, along SO, from the center of the sphere. We have the relation, $r^2 = R^2 - h^2$, and substituting the value of dh from this equation, we have as the resultant centrifugal moment

$$\sum_{0}^{R} \pi r^4 \delta \cdot \omega \sin \theta \dot{\psi} \, dh = \tfrac{4}{3} \pi R^3 \delta \cdot \tfrac{2}{5} R^2 \omega \sin \theta \dot{\psi}.$$

But $\frac{4}{3} \pi R^3 \delta$ is the mass of the sphere and $\frac{2}{5} R^2$ is k^2, k being the radius of gyration. Hence, the moment of momentum of the sphere about SO into the angular velocity $\dot{\psi}$ represents the resultant moment of the centrifugal forces about HH', or the gyroscopic couple, indifferently.

As the apparatus turns about the point S with the angular velocity $\dot{\psi}$, it is evident that the sphere is actually turning bodily about an axis $\perp$ to CC' with an angular velocity $\dot{\psi} \sin \theta$.

It is further evident that the component $\dot{\psi} \cos \theta$ of this turning does not in any way influence the rotation ω about the axis CC', since we have stipulated that there is no friction. The original rotational velocity remains constant, although the axes HH' and the axis $\perp$ to CC' and HH', which we shall hereafter call the $\dot{\psi} \sin \theta$ axis, both turn about CC' with an angular velocity $\dot{\psi} \cos \theta$. In the foregoing discussion we considered only the component $\omega \sin \theta$ about the axis SO. This was for the sake of simplicity. It will be seen that, resolving the rotation about the $\dot{\psi} \sin \theta$ axis on the SO axis, the total rotation about the SO axis

is $\omega \sin \theta - \dot{\psi} \sin \theta \cos \theta$, since these components are in opposite directions. We shall obtain, as before, that the sum of the centrifugal moments about HH' is $Mk^2 (\omega \sin \theta - \dot{\psi} \sin \theta \cos \theta)\, \dot{\psi} = C\omega \sin \theta\, \dot{\psi} - C\, \dot{\psi}^2 \sin \theta \cos \theta$, where C is the moment of inertia of the sphere. But we can write the same result at once from gyroscopic principles, thus saving ourselves a tedious inquiry into centrifugal forces. For the sphere is turning about the $\dot{\psi} \sin \theta$ axis with angular velocity $\dot{\psi} \sin \theta$, and this axis is being turned about the $\perp$ axis CC' with angular velocity $\dot{\psi} \cos \theta$. Hence there is set up a gyroscopic couple $C\dot{\psi} \sin \theta \cdot \dot{\psi} \cos \theta$. There is also a gyroscopic couple $C\omega \cdot \dot{\psi} \sin \theta$ acting in the opposite direction. Hence the resultant couple is

$$C\dot{\psi}^2 \sin \theta \cos \theta - C\omega\dot{\psi} \sin \theta = C\ddot{\theta}.$$

The rotational component about the vertical axis, or $\omega \cos \theta + \dot{\psi} \sin^2 \theta$, has no gyroscopic effect, since its axis already coincides with that of $\dot{\psi}$.

It can be easily shown that the resultant centrifugal force due to compounding the revolutional velocity with a rotational velocity about a vertical axis is $MD\dot{\psi}^2$ and is the same whether there is a rotation about a vertical axis or not. This is the expression for the revolutional centrifugal force and represents the stress on the rod SO, or the pressure on the point S. Nevertheless, these combined centrifugal forces in the horizontal plane produce a stress *in* the body, which it would not have if it were not rotating about a vertical axis. For it is clear that the actual velocities of the particles about S are, in the outer half of the sphere, the sums of the velocities due to rotation and revolution, while in the inner half they are the differences. We could thus, by integration, find a point in the outer half which we could call the center of centrifugal effort. On the other hand, the center of gravitational effort always lies in the

inner half, so that generally, in all planetary bodies, forces are developed which tend to pull the body apart in opposite directions from the center and along the line *SO*. An elastic sphere will therefore be deformed into an ellipsoid, with the long axis always pointing towards the attracting center. In the case of the earth it would seem that these solid tides are not inappreciable and may possibly amount to half a foot. The effect of the tides of the shallow, mobile oceans in slowing the earth's rotation is insignificant in comparison. Where a body rotates in an opposite direction to its revolution these solid tides would be greatly reduced and might be obliterated.

5.

We shall now take the general case of a tri-axial body having a rotation about some axis. We shall always suppose that the center of inertia is fixed. Such a rotation cannot in general be stable, for we can resolve the angular velocity about the momentary axis into its three components about the principal axes. Let ω_1 be the angular velocity at any instant about the axis A, ω_2 that about B and ω_3 that about C. The turning about B will set up a gyroscopic couple with the rotational moment about C, and so on reciprocally, so that there will be six gyroscopic couples — a pair about each axis. Taking the sum (algebraic) of each pair, and with due regard to signs, we can write

$$A\frac{d\omega_1}{dt} = (B - C)\,\omega_2\omega_3,$$

$$B\frac{d\omega_2}{dt} = (C - A)\,\omega_1\omega_3,$$

$$C\frac{d\omega_3}{dt} = (A - B)\,\omega_1\omega_2.$$

These are Euler's celebrated dynamical equations, which we have been able to write at once from the fundamental gyroscopic principle.*

It is evident that a rotation about any axis not a principal axis will set up gyroscopic couples which will cause the body to move away from that axis. But a rotation about a principal axis will be stable. Referring to Fig. 2, we see that the rotation was not stable because it was not about a principal axis, thereby causing the bar to oscillate about the horizontal plane. In the case of a sphere, where all axes are principal axes, any rotation will be stable.

6.

Let us suppose a sphere, Fig. 4, rotating about an axis OC with angular velocity ω. We now give it an impulsive angular velocity $\dot{\phi}$ about $OA \perp$ to OC. Let us say that we strike the end of the axis OC a sharp blow downward through the page. To find the motion. There will be an instantaneous rotation about a new axis, compounded of the other two rotations. If ι is the inclination of the new axis to

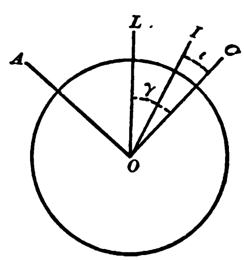

FIG. 4.

OC then $\tan \iota = \frac{\dot{\phi}}{\omega}$, and the angular velocity about the new axis is $\sqrt{\omega^2 + \dot{\phi}^2}$.

Now this first instantaneous rotation, being about a principal axis, will persist, and the motion will be stable.

* The usual demonstration is somewhat complicated. Some mathematicians (Staeckel) have expressed the opinion that the reasoning in the orthodox derivation is not wholly convincing. Euler himself (Berlin Memoirs, 1762) stated somewhat diffidently that he believed the equations were correct, but that he had no idea what the motion would be.

Our impulse, therefore, simply shifts the rotational axis to a new position between the other two, where it persists with a rotational velocity compounded of the other two velocities.

Let us suppose that the body is bi-axial, C and A having the usual significance. There will be an instantaneous new axis, as before, determined by the same conditions, viz., its angular velocity will be $\sqrt{\omega^2 + \dot{\phi}^2}$, and $\tan \iota = \frac{\dot{\phi}}{\omega}$, ι being the angle between the instantaneous axis and OC.

But the new axis is not a principal axis, and the motion cannot be stable. Consequently the instantaneous axis will move, and the plane COI must move also. Let us see if it is possible to find some line in this plane which does not move. Let OL be a line in this plane making an angle γ with OC. Let the plane turn about this line with angular velocity $\dot{\psi}$. Let us impose the conditions $A\dot{\psi} \cos \gamma = C\omega$ and $\dot{\psi} \sin \gamma = \dot{\phi}$. These two equations are sufficient to determine the two quantities $\dot{\psi}$ and γ, and only one definite value for each can satisfy the equations. Multiplying the first by $\dot{\psi} \sin \gamma$, we have $A\dot{\psi}^2 \sin \gamma \cos \gamma = C\omega\dot{\psi} \sin \gamma$. Now at the beginning of the impulse the conditions are these. The body is rotating about the axis OA with angular velocity $\dot{\phi} = \dot{\psi} \sin \gamma$. The plane $CILA$ is turning about OL with angular velocity $\dot{\psi}$. Hence the axis OA is turning about OC with angular velocity $\dot{\psi} \cos \gamma$. The original rotation ω about OC is uninfluenced. We have thus two gyroscopic couples, $A\dot{\psi} \sin \gamma \cdot \dot{\psi} \cos \gamma$ and $C\omega \cdot \dot{\psi} \sin \gamma$. But the conditions imposed make these equal, and as they are in opposite directions, there will be equilibrium about an axis $\perp$ to the plane. As the plane COI turns about OL, the angle γ must remain constant and likewise the angle ι. From the equations $C\omega = A\dot{\psi} \cos \gamma$ and $\frac{\dot{\phi}}{\omega} = \tan \iota$, it

follows that $A \tan \iota = C \tan \gamma$. Hence the motion is fully determined. It is the same as if a cone having OC for an axis and a half angle ι, fixed in the body, were rolling on a cone, fixed in space, having OL for its axis and a half angle $\gamma - \iota$. The axis OL is called the invariable line. A body moving in this manner, under the influence of no external forces, is said to perform a Poinsot motion. The motion of the plane COL about OL is called the precession. The velocity of the axes OI and OL relatively the body, in which they describe cones about OC, is evidently $\dot{\psi} \cos \gamma - \omega$. From the equation $C\omega = A\dot{\psi} \cos \gamma$ we have $C\omega \cos \gamma + A\dot{\psi} \sin^2 \gamma = A\dot{\psi}$, which means that the moment of momentum of the body about the invariable line remains constant and equal to $A\dot{\psi}$.

If instead of starting with a bi-axial body rotating about OC and then imparting an impulsive moment about OA, we start with the body at rest and then give it an impulsive moment about a line OL, equal to G, we may decompose this into two simultaneous impulsive moments about OC and OA, equal respectively to $G \cos \gamma$, and $G \sin \gamma$, where γ is the angle COL as before. If ω and $\dot{\phi}$ are the impulsive angular velocities about these axes, then $\tan \iota = \dfrac{\dot{\phi}}{\omega}$, and, from the above equations, $\tan \gamma = \dfrac{A}{C} \dfrac{\dot{\phi}}{\omega} = \dfrac{A}{C} \tan \iota$, and the case is the same as before. Hence we see that, in a bi-axial body, an impulse about any axis, not a principal axis, imparts an instantaneous rotation, not about that axis, but about another, found by the relation $A \tan \iota = C \tan \gamma$. The axis of the impulse will be the invariable line, and the moment of momentum about this line will remain constant and equal to G.

If the body be tri-axial and receive an impulsive moment about a line OL, not a principal axis, let α, β, γ be the angles

which the principal axes OA, OB, OC, respectively, make with OL, and let ω_1, ω_2, ω_3 be the angular velocities about these axes at any instant. We can decompose our impulse, G, into the three partial impulses,

$$G\cos\alpha = A\omega_1\ (1),\quad G\cos\beta = B\omega_2\ (2),\quad G\cos\gamma = C\omega_3\ (3).$$

Multiplying Euler's dynamical equations by ω_1, ω_2 and ω_3 we have

$$\frac{A\omega_1^2}{2} + \frac{B\omega_2^2}{2} + \frac{C\omega_3^2}{2} = T = \text{a constant.}$$

Or the kinetic energy remains constant throughout the motion. From (1), (2) and (3),

$$G = A\omega_1\cos\alpha + B\omega_2\cos\beta + C\omega_3\cos\gamma\ \ (4),\quad\text{and}$$
$$G^2 = A^2\omega_1^2 + B^2\omega_1^2 + C^2\omega_3^2.\ \ (5)$$

Let us suppose that the preponderating rotation is about OC, so that, as the body rotates about this axis, OA and OB become successively $\perp$ to OL. Since, from (4) the moment of momentum about OL remains constant and equal to G, it is evident that the moment about any other fixed line must remain constant, and the moment about any line $\perp$ to OL must be zero.

Hence, whenever OA or OB becomes $\perp$ to OL, ω_1 or ω_2 becomes zero. If OB is $\perp$ to OL we have, from (4) and (5),

$$\frac{A\omega_1^2}{2} + \frac{C\omega_3^2}{2} = T,$$
$$A\omega_1\sin\gamma + C\omega_3\cos\gamma = G,$$
$$A^2\omega_1^2 + C^2\omega_3^2 = G^2.$$

These three equations determine ω_1, ω_3 and γ. In the same manner we derive three other values for ω_2, ω_3 and γ when OA is $\perp$ to OL, and it is evident that at these instants the motion would coincide with that of a bi-axial

body under the same impulse. It will be seen, therefore, that a tri-axial body, subjected to an impulse about a line *OL*, will perform an irregular precession about that line, with the axis *OC* approaching and receding from that line within definite limits, being nearest to it when *OB* is ⊥ to *OL* and farthest from it when *OA* is ⊥ to *OL*.

To recapitulate: When we impart an impulsive couple to a uni-axial body it will, from the first, rotate about the axis of the couple with a constant velocity. The axis of the couple, the instantaneous axis and the invariable line are one. If the body be bi-axial, the instantaneous axis does not coincide with the impulse axis (unless this be a principal axis) and the motion consists of a constant rotation about *OC*, which is carried about the impulse axis (or invariable line) with a constant precessional velocity at a constant distance from the invariable line. The instantaneous axis always remains in the precessional plane at a constant distance from the invariable line.

If the body be tri-axial, the motion consists of a rotation about a principal axis, which is not constant, but varies within fixed limits, while this axis is carried around the invariable line with a precessional velocity which varies within fixed limits, being greatest when *OB* is ⊥ to *OL* and least when *OA* is ⊥ to *OL*. This axis, further, does not preserve a constant distance from the invariable line (or impulse axis), but moves in the precessional plane within fixed limits. The instantaneous axis does not remain in the precessional plane (*COL*), but moves to either side of it, within fixed limits, being in this plane only when *OA* or *OB* is ⊥ to *OL*. We may describe the motion, therefore, as a varying rotation about a principal axis, while this axis performs an irregular or wobbly precession about the invariable line. None of the elements of motion are constant, but vary within fixed limits. In all three

cases, however, the axis of the impulse couple is the invariable line. Such motions of a body about a fixed point, started by an impulsive couple and without the action of any other forces, are called Poinsot motions.* For a triaxial body, the path of C, the extremity of the main rotation axis, will be a symmetrical wavy curve about the invariable line, like that represented in Fig. 13.

The mere translational motion of a body, or motion in which the axes remain ||, can set up no gyroscopic action. Hence the translational motion of the center of inertia, and rotations about that center, are independent. So that, generally, whenever an irregular body is struck, we can resolve the impulse into an impulsive velocity of the center of gravity and an impulsive couple about that center, and the center of inertia will describe its path just the same as if the body were not rotating, while the body will execute a Poinsot motion about that center, just the same as if it were fixed. Motion of the first kind we are familiar with and can easily anticipate, while motion of the second kind, or gyroscopic motion, is contrary to our everyday experience and cannot be foreseen. It can only be brought out by analysis.†

We shall refer once more to the popular superstition that a gyroscope "tends to maintain its plane of rotation," or, in other words, that it is a "gyrostat." If it is understood by this that a gyroscope opposes to a change of plane what we have called rotational inertia, just as a simple body opposes ordinary inertia to a change of position, then no great harm is done. But if it is supposed that a gyroscope is a gyrostat, viz., that it maintains its plane, then grievous

* Poinsot. Theorie Nouvelle de la Rotation des Corps, 1834.

† Gilbert, in one of his operas, humorously refers to the terrible punishment meted out to a billiard fiend, who was condemned to play with "elliptical billiard balls."

harm may result. A gyroscope spinning with an infinite velocity — a mathematical fiction — cannot have its plane changed by any finite force. In this case, it *is* a gyrostat; but finitely spinning gyroscopes do not and cannot maintain their planes when acted upon by an outside couple. A misunderstanding of this fact has led to many strange mechanical attempts. The late Sir Henry Bessemer, of steel fame, misled by the name gyrostat, actually constructed a cabin on a ship, swung on fore and aft trunnions, to which was rigidly attached a heavy rotating flywheel. The idea was that the "gyrostat" would maintain its plane and with it the cabin, so that the ship might roll, but not the cabin. Of course the cabin swung just the same as if there had been no rotating flywheel attached to it. The only result was that at each swing a heavy couple was brought to bear on the bearings of the wheel, tending to stop its rotation by the increased friction. If he had allowed the axle to move relatively to the cabin — to precess — then a slight swing would have set up a precessional reaction — our rotational inertia would have come into play — and he might have anticipated Schlick's Stabilisator.

As another example of the prevalent misconception that a gyroscope is a gyrostat, we find the following in Prof. Sylvanus Thompson's classical work on "Dynamo-electric Machinery." "Another point, which arises only in the case of dynamos used on shipboard and motors running round a curve on a track, is the *gyrostatic action* of the revolving armature, which always tends to keep its axis pointing in the same direction." He then states that this gives rise to a force "On each bearing, alternately acting *up* and *down* at each roll, if the axis of the dynamo lies athwart the ship." There is, of course, no such thing as gyrostatic action. There is, however, in this case, a gyroscopic couple, which does not act *up* and *down*, but back-

wards and forwards, i.e., about an axis ⊥ to the deck. From this we see that it is advisable, as far as possible, to place the axes of all rotating masses on a ship in a fore-and-aft direction, since, in such a position, the only gyroscopic couples set up will be those due to the pitching of the ship.

7. External Forces.

We shall now investigate the motions of gyroscopes under the action of external forces. Let Fig. 5 represent

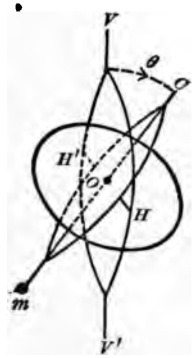

FIG. 5.

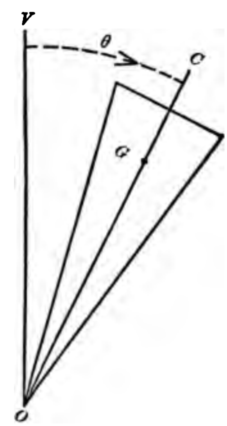

FIG. 6.

a bi-axial ellipsoid rotating with angular velocity ω about its axis of greatest moment OC. This axis is held in a ring which is pivoted about a horizontal axis HH', in a vertical frame VV'. This vertical frame can turn about the vertical axis VV'. To the lower extremity of the rotation axis, at a distance l from O, is attached a small mass m. We shall suppose this small mass and the supporting frame to be negligible in comparison with the mass of the gyroscope. Fig. 6 represents a bi-axial top, its point fixed at O, and its

center of gravity at G, OG being represented by l. To find the motion. The two problems are identical, with the exception that, in one case, the axis OC is urged upward, while in the other case it is urged downward. Both gyroscopes start from rest and are influenced by gravity. The rotations are both positive viewed from above.

The gravitational couple is $\pm\, mgl \sin\theta$. It is clear that any velocity $\dot\theta$ will set up a gyroscopic couple about an axis $\perp$ to OC and in the vertical plane COV. We have called this the $\dot\psi \sin\theta$ axis. The horizontal axis HH' we shall call the $\dot\theta$ axis. $\dot\psi$ represents the angular velocity about the vertical OV — or the precessional velocity. It will be noted that the $\dot\psi \sin\theta$ axis and the $\dot\theta$ axis both turn about OC with angular velocity $\dot\psi \cos\theta$, while the rotations about the three mutually $\perp$ axes we have taken are ω, $\dot\psi \sin\theta$ and $\dot\theta$. We can therefore write the equations of motion at once, and they are (Fig. 6)

$$mgl\sin\theta - C\omega\dot\psi\sin\theta + A\dot\psi^2\sin\theta\cos\theta = A\ddot\theta, \qquad (1)$$

$$C\omega\dot\theta - A\dot\psi\cos\theta\dot\theta = AD_t(\dot\psi\sin\theta). \qquad (2)$$

Multiplying (2) by $\sin\theta$,

$$C\omega\sin\theta\dot\theta - A\dot\psi\sin\theta\cos\theta\dot\theta = A\sin\theta D_t(\dot\psi\sin\theta). \qquad (3)$$

Integrating,

$$C\omega(\cos\theta_0 - \cos\theta) - A\int\dot\psi\sin\theta\cos\theta\dot\theta = A\dot\psi\sin^2\theta$$
$$- A\int\dot\psi\sin\theta\cos\theta\dot\theta,$$

or, $$A\dot\psi\sin^2\theta + C\omega\cos\theta = C\omega\cos\theta_0, \qquad (4)$$

where θ_0 is the initial inclination.

Equation (4) informs us that the moment of momentum about the vertical remains constant, as was à priori evident,

and we might have written the equation at once without integrating. It is evident that there is no couple acting about the vertical and therefore no increase (or decrease) of the original moment of momentum about that axis can occur. And, generally, it is evident that the time integral of any outside couple acting about a fixed axis must equal the increase (or decrease) of the moment of momentum about that axis. Gyroscopic couples, which are centrifugal forces and which we may call *internal* forces, can perform no work on the body. For action is always equal to reaction, and to perform work it is necessary to have an *external* "point d'appui." Work can only be done by external forces, and in our subsequent work we shall see that gyroscopic forces never perform work on the body. Gyroscopic couples react mutually on each other and can no more perform work than a man can raise himself by his boot straps. But gyroscopic forces change the *direction* of the motion caused by external forces and, as we shall see, in a most peculiar manner.

Multiplying equation (1) by $\dot{\theta}$ and equation (2) by $\dot{\psi}\sin\theta$, and then adding and integrating, we have

$$mgl(\cos\theta_0 - \cos\theta) = \frac{A\dot{\theta}^2}{2} + \frac{A\dot{\psi}^2\sin^2\theta}{2}. \qquad (5)$$

That is, the kinetic energy imparted is strictly equal to the external work done by gravitation. This was also à priori evident, and might have been written at once.

From equation (4) we have

$$\dot{\psi} = \frac{C\omega(\cos\theta_0 - \cos\theta)}{A\sin^2\theta}.$$ Substituting in equation (5)

$$mgl(\cos\theta_0 - \cos\theta) = \frac{A\dot{\theta}^2}{2} + \frac{[C\omega(\cos\theta_0 - \cos\theta)]^2}{2A\sin^2\theta}. \qquad (6)$$

Whence

$$dt = \frac{\sin\theta\, d\theta}{\sqrt{\frac{2\,mgl}{A}(\cos\theta_0 - \cos\theta)\,(1-\cos^2\theta) - \left[\frac{C\omega}{A}(\cos\theta_0 - \cos\theta)\right]^2}}.$$

This is an elliptic integral which can be readily solved, though the transformation is somewhat tedious. We can thus find the inclination of the axis at any instant, and from equation (4) the value of $\dot{\psi}$ at that instant. The motion is completely determined.

We are, however, not so much concerned with the exact quantitative determination of the path described by the axis, as with its quality, or what kind of motion the axis undergoes. Let us see if the axis falls continuously, or if there is a limit below which it cannot go.

Putting $\dot{\theta} = o$ in equation (6), we have

$$\frac{2\,mgl}{A}(1 - \cos^2\theta) = \frac{(C\omega)^2}{A}(\cos\theta_0 - \cos\theta). \qquad (7)$$

Whence

$$\cos\theta = \frac{(C\omega)^2}{4\,mglA} \pm \sqrt{1 - \frac{(C\omega)^2\cos\theta_0}{2\,mglA} + \frac{(C\omega)^4}{16\,m^2g^2l^2A^2}}.$$

The positive sign leads to a value for $\cos\theta$, greater than unity, and is therefore inadmissible. Hence the axis will cease falling (or rising) at the point where

$$\cos\theta = \frac{(C\omega)^2}{4\,mglA} - \sqrt{1 - \frac{(C\omega)^2\cos\theta_0}{2\,mglA} + \frac{(C\omega)^4}{16\,m^2g^2l^2A^2}}. \quad (8)$$

The nature of the motion is now clear. At a certain point, the axis stops falling, and all the kinetic energy is converted into horizontal motion. This carries it up to the inclination from which it fell, when, all the kinetic energy being used up, it comes momentarily to rest again, only to repeat the process over and over again. From

equation (4), the horizontal motion or precession must always be positive. Hence the axis keeps moving about the vertical in one direction, executing meanwhile a series of symmetrical dips. If ω is very large, we see from equation (7) that the dips, or $(\cos\theta_0 - \cos\theta)$, must be very small, and from equation (4) that the horizontal velocity, or $\dot{\psi}$, must also be very small. $\dot{\theta}$ is accordingly very small, and the squares and products of these very small quantities can be neglected in comparison with the quantities themselves.

If ω is very large, we can, therefore, write equations (1) and (2),

$$mgl\sin\theta - C\omega\dot{\psi}\sin\theta = A\ddot{\theta}. \qquad (9)$$

$$C\omega\dot{\theta} = AD_t(\dot{\psi}\sin\theta). \qquad (10)$$

Now the very small portion of a sphere included by one of the dips, we can regard as practically plane, and the quantities $\psi\sin\theta$ and θ become rectangular coördinates, $\psi\sin\theta$ being represented by x, θ by y. The origin of coordinates is where the motion began, viz., from rest.

We can further regard $\sin\theta$ as practically constant in this small area. Hence equations (9) and (10) become

$$mgl\sin\theta - C\omega\dot{x} = A\ddot{y}. \qquad (11)$$

$$C\omega\dot{y} = A\ddot{x}. \qquad (12)$$

Integrating,

$$mgl\sin\theta t - C\omega x = A\dot{y}. \qquad (13)$$

$$C\omega y = A\dot{x}. \qquad (14)$$

Writing the equations

$$x = \frac{mgl\sin\theta A}{C^2\omega^2}\left[\frac{C\omega t}{A} - \sin\left(\frac{C\omega}{A}t\right)\right] \qquad (15)$$

$$y = \frac{mgl\sin\theta A}{C^2\omega^2}\left[1 - \cos\left(\frac{C\omega}{A}t\right)\right] \qquad (16)$$

we see that they are the integrals of (14) and (13) respec-

tively. For differentiating (15) and (16) and substituting the values of x and y, we obtain (14) and (13).

Equations (15) and (16) represent a cycloid with its base horizontal, convex downward, and having a cusp at the origin of coördinates. The equations of a cycloid are usually written $x = a(\phi - \sin\phi)$, $y = a(1 - \cos\phi)$, where a is the radius of the generating circle and ϕ is the angle which a radius makes at any time with its initial position.

Hence our cycloid is generated by a circle of radius $\frac{mgl \sin\theta A}{C^2\omega^2}$, and the angle which any radius makes with its initial position is proportional to the time and equal to $\frac{C\omega}{A}t$. If, therefore, we supposed our axis to be attached to the circumference of a very small wheel, of radius $\frac{mgl \sin\theta A}{C^2\omega^2}$, rolling along the under side of a parallel of latitude with the uniform angular velocity $\frac{C\omega}{A}$, it would describe the path taken by the extremity of the axis, both in time and position. The axis will thus describe in space a cycloidally fluted cone about the vertical as an axis. The time of falling through one of these minute cycloids is $\frac{2\pi A}{C\omega}$.

The rise and fall is so minute and rapid that the eye cannot follow it, or at most detects only a slight blurring. The ear, however, can detect the humming caused by these minute vibrations. When a gyroscope is turning about its point of support, a distinct note is usually heard, and the frequency of the vibrations can be very accurately determined by comparing it with the note of a tuning fork. Unless the gyroscope is driven by electricity or some other constant source, and thus kept up to pitch, it will be noticed

that the note constantly gets lower, corresponding to the slowing down of the rotation due to friction.

The axis progresses along the parallel of latitude a distance $2\pi\dfrac{mgl\sin\theta A}{C^2\omega^2}$, or through one cycloid, in the time $\dfrac{2\pi A}{C\omega}$. Hence the time of a complete revolution about the vertical will be $\dfrac{2\pi C\omega}{mgl}$. The times of a single vibration and of a complete revolution are, therefore, independent of the original inclination, and while the time of a single vibration is independent of the outside force, that of the revolution is inversely proportional to it.

The horizontal motion, or motion in longitude, is called the *precession*, while the variations of inclination, or latitude, are called the *nutations*. The nutations we have just investigated are called free nutations. We shall shortly come upon another class known as forced nutations.

If the gyroscope (Fig. 6) swings about the point O, below the horizontal plane through O, it becomes a gyroscopic pendulum. If ω is small, it will execute a series of festoons about the vertical, but it will never reach the nadir. If ω becomes zero, the body becomes a simple pendulum, and, from equation (6), we have

$$t = \int\frac{d\theta\sqrt{A}}{\sqrt{2\,mgl\,(\cos\theta_0 - \cos\theta)}},$$

the law of the pendulum. If we constrain the end of the axis to move in a frictionless groove in a vertical plane, the gyroscope falls like a simple pendulum, for, in this case, the gyroscopic couple called into play exhibits itself solely in pressure against the constraint, and the only θ acceleration is that due to gravity.

8. Case of Constant Precessional Velocity.

Let us suppose, Fig. 7, a ring which can turn about a horizontal axis HH'. Within this ring is pivoted another ring capable of turning about an axis PP', $\perp$ to HH'. This inner ring carries a rotating disc with its axis OC $\perp$

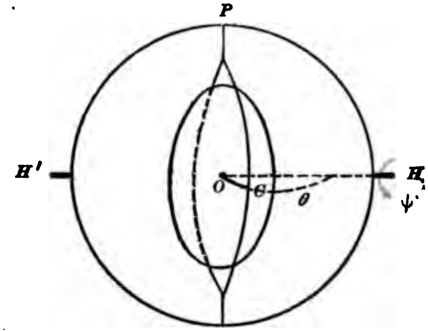

FIG. 7.

to PP'. A constant angular velocity, $\dot{\psi}$, is given to the outer ring about HH'. Rotations are positive and the angle HOC is θ. It is evident that the axis OC will move to set itself into coincidence with HH', that it will go beyond to an equal distance and then come back, thus oscillating about OH. It is in fact a horizontal pendulum, with the directive couple $C\omega\dot{\psi}\sin\theta - A\dot{\psi}^2\sin\theta\cos\theta$, tending to set the rotational axis along HH'.

We have at once the equation

$-C\omega\dot{\psi}\sin\theta + A\dot{\psi}^2\sin\theta\cos\theta = A\ddot{\theta}$, where A is the moment of inertia about an axis $\perp$ to OC.

Let the initial inclination be θ_0.

Hence, integrating,

$$C\omega\dot{\psi}(\cos\theta - \cos\theta_0) + A\dot{\psi}^2\left(\frac{\sin^2\theta}{2} - \frac{\sin^2\theta_0}{2}\right) = \frac{A\dot{\theta}^2}{2}$$

and $$\dot{\theta} = \sqrt{\frac{2\,C\omega\dot{\psi}}{A}(\cos\theta - \cos\theta_0) - \dot{\psi}^2(\cos^2\theta - \cos^2\theta_0)}.$$

The integration of this elliptic integral is, as usual, rather tedious. Let us suppose that $\dot{\psi}$ is very small compared with ω. We may then neglect its square without great error. Writing $\dfrac{C\omega\dot{\psi}}{A} = a$, we have

$$dt = \frac{d\dfrac{\theta}{2}}{\sqrt{a\left(\sin^2\dfrac{\theta_0}{2} - \sin^2\dfrac{\theta}{2}\right)}}.$$

Let $\sin\dfrac{\theta_0}{2}\sin\phi = \sin\dfrac{\theta}{2}$, where ϕ is an auxiliary angle. It will be seen that when $\theta = 0$, $\phi = 0$; and when $\theta = \theta_0$, $\phi = \dfrac{\pi}{2}$. Making this substitution,

$$dt = \frac{1}{\sqrt{a}}\cdot\frac{d\phi}{\sqrt{1 - \sin^2\dfrac{\theta_0}{2}\sin^2\phi}}.$$

The radical can be developed by the binomial theorem, and

$$\frac{1}{\sqrt{1 - \sin^2\dfrac{\theta_0}{2}\sin^2\phi}} = 1 + \frac{1}{2}\sin^2\frac{\theta_0}{2}\sin^2\phi$$

$$+ \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\sin^4\frac{\theta_0}{2}\sin^4\phi + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\sin^6\frac{\theta_0}{2}\sin^6\phi \ldots \text{ etc.}$$

Now $$\int_0^{\frac{\pi}{2}} \sin^{2n}\phi\, d\phi = \frac{1\cdot 3\cdot 5 \ldots\ldots 2n-1}{2\cdot 4\cdot 6 \ldots 2n}\cdot\frac{\pi}{2}.$$

Hence

$$t = \frac{\pi}{2\sqrt{a}}\left[1 + \left(\frac{1}{2}\right)^2\sin^2\frac{\theta_0}{2} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\sin^4\frac{\theta_0}{2}\right.$$

$$\left. + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\sin^6\frac{\theta_0}{2} \ldots \text{ etc.}\right]$$

where t is the time taken by the axis OC to swing from θ_0 to coincidence with OH. The time of a complete swing is

$$T = 2\pi\sqrt{\frac{A}{C\omega\dot{\psi}}}\left[1 + \left(\frac{1}{2}\right)\sin^2\frac{\theta_0}{2} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\sin^4\frac{\theta_0}{2}.\ .\ .\ .\ \text{etc.}\right]$$

If we attach the outer ring firmly to any point of the earth's surface with the axis PP' vertical, it is evident that the horizontal component of the earth's rotation, about an axis in the meridian, viz. $\dot{\psi}\cos\lambda$, where λ is the latitude of the place, will cause the positive end of the axis of the gyroscope to oscillate about the true North. The vertical component, $\dot{\psi}\sin\lambda$, will, of course, have no effect on the gyroscope. The gyroscope can thus be used as a compass to indicate the true North. Or if the axis PP' were set horizontally, due East and West, the axis of the gyroscope would oscillate about a direction ∥ to the earth's axis and, coming to rest in this position by damping, would indicate, by its elevation above the horizon, the latitude of the place. The rotation $\dot{\psi}\cos\lambda$ in the case of the earth is small enough for us to neglect its square in comparison with a high value of gyroscopic rotation. Hence, for small oscillations, we can use the formula

$$T = 2\pi\sqrt{\frac{A}{C\omega\dot{\psi}\cos\lambda}}.$$

Since for any disc, $\frac{A}{C} = \frac{1}{2}$, if the gyroscope has a rotation of 20,000 turns per minute, a value which is reached in practice, the period would be about 11 seconds, at the equator.

$$\dot{\psi} = \frac{2\pi}{86{,}164.1\text{ sec.}} = 0.0000729211.$$

$$\omega = 2\pi \times 333.33 = 2094 \text{ radians per second.}$$

9. Case of Constant Couple about a Fixed Axis.

Let a gyroscope be subjected to a constant couple about an axis which we shall take as the vertical. All symbols have the usual significance. Let H be the constant couple about the vertical.

We shall suppose H to be negative, so that, if the angle θ be less than $\frac{\pi}{2}$, ω, the original rotational velocity about OC, will be retarded.

The equations of motion are

$$-H\cos\theta = CD_t\omega_3. \tag{1}$$

$$-C\omega_3\dot\psi\sin\theta + A\dot\psi^2\sin\theta\cos\theta = A\ddot\theta. \tag{2}$$

$$-H\sin\theta + C\omega_3\dot\theta - A\dot\psi\cos\theta\dot\theta = AD_t(\dot\psi\sin\theta). \tag{3}$$

Multiplying (1) by $\cos\theta$ and (3) by $\sin\theta$, and adding,

$$-H = C\cos\theta D_t\omega_3 - C\omega_3\sin\theta\dot\theta + A\dot\psi\sin\theta\cos\theta\dot\theta$$
$$+ A\sin\theta D_t(\dot\psi\sin\theta).$$

Integrating,

$$-Ht = C\omega_3\cos\theta + A\dot\psi\sin^2\theta - C\omega\cos\theta_0. \tag{4}$$

This is the momental equation, which states that the time integral about the vertical axis is the increase of the moment of momentum about that axis.

Multiplying (2) by $\dot\theta$, and (3) by $\dot\psi\sin\theta$, and adding,

$$-H\dot\psi\sin^2\theta = A\ddot\theta\dot\theta + AD_t(\dot\psi\sin\theta)\cdot\dot\psi\sin\theta.$$

Integrating,

$$-\int H\sin\theta\cdot\sin\theta\,d\psi = \frac{A\dot\theta^2}{2} + \frac{A\dot\psi^2\sin^2\theta}{2} = T. \tag{5}$$

This equation states the fact that the work done by the couple about the $\dot\psi\sin\theta$ axis is equal to the kinetic energy about the $\dot\psi\sin\theta$ and $\dot\theta$ axes, or to the kinetic energy outside of that about the rotational axis, which, of course, it cannot influence.

Substituting from (1), $\cos\theta = -\frac{C}{H}D_t\omega_3$, in (4),

$$-Ht = -\frac{C^2}{H}\omega_3 D_t\omega_3 - C\omega\cos\theta_0 + A\dot{\psi}\sin^2\theta. \quad (6)$$

Integrating, and substituting the value of $\int\dot{\psi}$ $\sin^2\theta$ from (5), we have

$$\frac{H^2t^2}{2A} = \frac{C^2}{2A}(\omega_3^2 - \omega^2) + \frac{HC\omega}{A}\cos\theta_0 t + T. \quad (7)$$

Let us suppose that the couple H is small relatively to the initial rotation ω, and, therefore, ω_3, for a short time, does not differ greatly from ω. Eliminating t from equation (7) by the aid of equation (4), we find that the equation contains only the square of $\dot{\theta}$, while it has the first and second powers of $\dot{\psi}\sin\theta$. Hence when $\dot{\psi}\sin\theta$ becomes zero, the two values of $\dot{\theta}$ are equal but of opposite sign, while, when $\dot{\theta}$ becomes zero, we have two unequal values for $\dot{\psi}\sin\theta$.

It is evident that the rotation axis will at first move in obedience to the couple, but will shortly be deflected downward, executing a series of loops, as in Fig. 8.

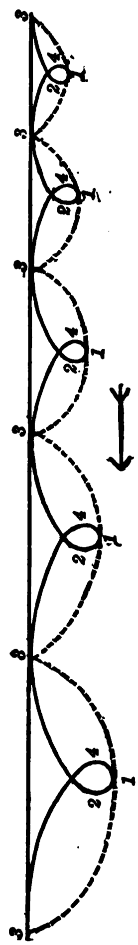

FIG. 8.

Let us take the case where the axis, at the beginning of motion, was $\perp$ to the axis of the couple, and, as before, the couple is so small, relatively to ω, that we may consider ω_3 as reasonably constant for a short time, and equal to ω.

Equation (7) becomes $\frac{H^2t^2}{2A} = \frac{A\dot{\theta}^2}{2} + \frac{A\dot{\psi}^2\sin^2\theta}{2}$.

Substituting from (4), $\dot{\psi} \sin \theta = \dfrac{-Ht - C\omega \cos \theta}{A \sin \theta}$,

we have
$$\frac{H^2t^2}{2A} = \frac{A\dot{\theta}^2}{2} + \frac{1}{2A} \cdot \frac{(Ht + C\omega \cos \theta)^2}{\sin^2 \theta}. \tag{8}$$

Let us now determine the *locus* where $\dot{\theta}$ becomes zero. We do this by putting $\dot{\theta} = 0$ in equation (8).

Hence
$$Ht = \pm \frac{(Ht + C\omega \cos \theta)}{\sin \theta}, \text{ or}$$

$$Ht (\sin \theta \mp 1) = \pm C\omega \cos \theta. \tag{9}$$

Measuring the angle θ from the equator instead of from the pole, $\cos \theta$ becomes $-\sin \theta_1$, and $\sin \theta$ becomes $\cos \theta_1$, where θ_1 is a very small angle. Hence equation (9) becomes $Ht (\cos \theta_1 \mp 1) = \mp C\omega \sin \theta_1$. Since the equation must vanish, identically, when $\theta_1 = 0$, the lower signs are inadmissible. Putting $\sin \theta_1 = y$, we have

$$y = \frac{Ht}{C\omega}(1 - \cos \theta_1). \tag{10}$$

This is the familiar equation of a cycloid with a generating circle of radius $\dfrac{Ht}{C\omega}$. That is, the radius of the generating circle increases proportionately to the time. The points where $\dot{\theta}$ becomes zero all lie, therefore, on this cycloid as a locus. The cycloids are represented by the dotted curve in Fig. 8, and the looped curve of the axis touches these cycloids at the points 3, 1, while the points 2, 4, indicate where $\dot{\psi} \sin \theta$ becomes zero, $\dot{\theta}$ having equal but opposite (in sign) values at these points.

The proportions are, of course, very much exaggerated in the figure. Actually, with a very large value for ω, the loops would be very minute and the axis would move along close to the equator for an appreciable time. It would,

of course, finally get away, for, as the excursions become larger, the value of ω begins to increase and our equation no longer holds. The general motion would be that, while executing a looped path, it would gradually spiral downward, pass through the nadir, rise again, though not to its original level, and thus, by a series of swings, finally come to rest at the nadir, where the couple would be expended solely in increasing the rotational energy about OC.

10. Attraction of a Distant Body on a Tri-axial Body.

Let O, Fig. 9, be the center of inertia of the attracted tri-axial body, OC the axis of greatest moment, OA that of

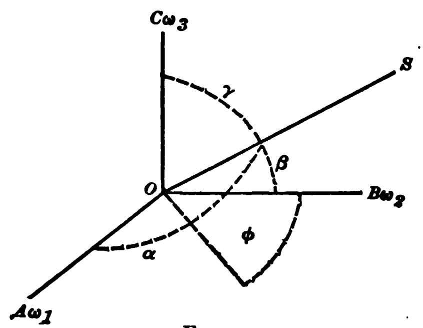

FIG. 9.

least moment and OB that of middle moment. Let OS be the direction of the attracting body, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ the direction cosines of the axes OA, OB, OC, respectively with respect to OS. Let ϕ be the angle between the planes SOC and BOC, and the initial rotation about OC, ω, while ω_1, ω_2, ω_3 are the rotations about the axes at any instant. The initial rotations about OA and OB are supposed zero.

It is proved in treatises on Dynamics that the total attraction of a distant body on such a tri-axial body can

be resolved into a simple attraction on the mass supposed to be concentrated into the center of inertia, and three couples about the three principal axes. Hence if we suppose the simple attraction to be balanced by the revolutional centrifugal force of the body, the problem reduces itself to finding the motion about its center, which we can suppose to be fixed.

We shall not take the time here to give the proof, since it can be found in any text book,* but simply state that the couples about the axes OA, OB, OC are, respectively,

$$\frac{3S}{D^3}(C - B)\cos\beta\cos\gamma,$$

$$\frac{3S}{D^3}(A - C)\cos\alpha\cos\gamma,$$

$$\frac{3S}{D^3}(B - A)\cos\alpha\cos\beta,$$

where S is the mass of the attracting body and D its distance from O. Let $K = \frac{3S}{D^3}$

We can write at once the equations of motion,

$$(C - B)\,[K\cos\beta\cos\gamma - \omega_2\omega_3] = AD_t\omega_1. \qquad (1)$$

$$(A - C)\,[K\cos\alpha\cos\gamma - \omega_1\omega_3] = BD_t\omega_2. \qquad (2)$$

$$(B - A)\,[K\cos\alpha\cos\beta - \omega_1\omega_2] = CD_t\omega_3. \qquad (3)$$

The terms containing K express the attractional couples about the respective axes, while the terms containing the ω's express the gyroscopic couples about these axes. Multiplying equation (1) by ω_1, equation (2) by ω_2, and equation (3) by ω_3, and adding,

$$K\,[(C - B)\cos\beta\cos\gamma\omega_1 + (A - C)\cos\alpha\cos\gamma\omega_2 + (B - A)\cos\alpha\cos\beta\omega_3]$$
$$= A\omega_1D_t\omega_1 + B\omega_2D_t\omega_2 + C\omega_3D_t\omega_3. \text{ Or,}$$

* Routh. Advanced Dynamics, Art. 519.

$$K\int[(C-B)\cos\beta\cos\gamma\omega_1+(A-C)\cos\alpha\cos\gamma\omega_2$$
$$+(B-A)\cos\alpha\cos\beta\omega_3]$$
$$=A\frac{\omega_1^2}{2}+B\frac{\omega_2^2}{2}+C\frac{(\omega_3^2-\omega^2)}{2}. \tag{4}$$

The left member is the work done by the gravitational couples, while the right member is the imparted kinetic energy. It will be seen that the work done by the gyroscopic couples is zero. (See Art. 7.)

From a simple geometrical consideration, it is readily seen that $\cos\phi=\dfrac{\cos\beta}{\sin\gamma}$, and $\sin\phi=\dfrac{\cos\alpha}{\sin\gamma}$, and that $\dot\gamma=\omega_1\cos\phi-\omega_2\sin\phi$. Whence

$$\left.\begin{aligned}\sin\gamma\dot\gamma&=\omega_1\cos\beta-\omega_2\cos\alpha,\\ \sin\beta\dot\beta&=\omega_3\cos\alpha-\omega_1\cos\gamma,\\ \sin\alpha\dot\alpha&=\omega_2\cos\gamma-\omega_3\cos\beta.\end{aligned}\right\}\text{and by symmetry} \tag{5}$$

Multiplying equation (1) by $\cos\alpha$, equation (2) by $\cos\beta$, and equation (3) by $\cos\gamma$, and adding,

$$\left.\begin{aligned}&-(C-B)\,\omega_2\omega_3\cos\alpha,\\ &-(A-B)\,\omega_1\omega_3\cos\beta,\\ &-(B-A)\,\omega_1\omega_2\cos\gamma.\end{aligned}\right\}=A\cos\alpha\,D_t\omega_1+B\cos\beta D_t\omega_2+C\cos\gamma D_t\omega_3.$$

Or,

$$\left.\begin{aligned}&-A\omega_1\omega_2\cos\gamma+A\omega_1\omega_3\cos\beta+A\cos\alpha D_t\omega_1\\ &-B\omega_2\omega_3\cos\alpha+B\omega_1\omega_2\cos\gamma+B\cos\beta D_t\omega_2\\ &-C\omega_1\omega_3\cos B+C\omega_2\omega_3\cos\alpha+C\cos\gamma D_t\omega_3\end{aligned}\right\}=0. \tag{6}$$

Now the first line is by (5) the derivative of $A\omega_1\cos\alpha$, the second line is the derivative of $B\omega_2\cos\beta$, and the third line is the derivative of $C\omega_3\cos\gamma$.

Hence, integrating equation (6), we have

$$A\omega_1\cos\alpha+B\omega_2\cos\beta+C\omega_3\cos\gamma=C\omega\cos\gamma_0. \tag{7}$$

This expresses the fact that the moment of momentum about OS remains constant, as was à priori evident.

$D_t A\omega_1 \cos\alpha =$ [illegible] $\dot\omega_1 +$ [illegible] $\sin\alpha\,\dot\alpha\}$
$= A\cos\alpha\,\dot\omega_1 -$ [illegible] $\omega_2 \cos\gamma +$ [illegible]

From (5), we have

$$\cos\beta\cos\gamma\omega_1 = \sin\gamma\cos\gamma\dot{\gamma} + \cos\alpha\cos\gamma\omega_2.$$
$$\cos\alpha\cos\beta\omega_3 = -\sin\alpha\cos\alpha\dot{\alpha} + \cos\alpha\cos\gamma\omega_2.$$

Multiplying (1), (2), (3) by ω_1, ω_2 and ω_3, and substituting these values, we have

$$K(C-B)\sin\gamma\cos\gamma\dot{\gamma} - K(B-A)\sin\alpha\cos\alpha\dot{\alpha}$$
$$= A\omega_1 D_t\omega_1 + B\omega_2 D_t\omega_2 + C\omega_3 D_t\omega_3.$$

Integrating,

$$K[(C-B)(\sin^2\gamma - \sin^2\gamma_0) - (B-A)(\sin^2\alpha - \sin^2\alpha_0)]$$
$$= A\omega_1^2 + B\omega_2^2 + C(\omega_3^2 - \omega^2), \tag{8}$$

which we can write in the symmetrical form

$$K[C(\sin^2\gamma - \sin^2\gamma_0) + B(\sin^2\beta - \sin^2\beta_0) + A(\sin^2\alpha - \sin^2\alpha)]$$
$$= A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \tag{9}$$

The work done is therefore expressed by half the left member.

We have further by combining equations (1), (2) and (3) in pairs,

$$\frac{A}{C-B}\omega_1^2 + \frac{B}{C-A}\omega_2^2 = K(\sin^2\gamma - \sin^2\gamma_0), \tag{10}$$

$$\frac{B}{C-A}\omega_2^2 + \frac{C}{B-A}(\omega_3^2 - \omega^2) = K(\sin^2\alpha_0 - \sin^2\alpha), \tag{11}$$

$$\frac{A}{C-B}\omega_1^2 + \frac{C}{B-A}(\omega^2 - \omega_3^2) = K(\sin^2\beta_0 - \sin^2\beta). \tag{12}$$

These are the kinetic equations. The earth being a tri-axial body, although only slightly deviating from bi-axiality, its rotation, ω_3, is not constant, but variable. It is possible that this irregularity, slight as it is, might be determined astronomically, and the positions of the axes A and B, as well as their ratio, approximated. This would be a great improvement over the laborious methods of

direct measurement, employed by Col. Clarke and others. Col. Clarke's first results were that the longer equatorial axis exceeded the shorter by about a mile, and that this axis was situated in longitude 15° 34′ E. Later, he changed this to 8° 15′ W. The uncertainty shows the futility of using minutes in the results.

The general solution of the motion of a tri-axial body under external forces, has not hitherto been effected. [See Note at end.]

11. Attraction of Distant Body on a Bi-axial Body.

In the case of a bi-axial body, we have as the equations of motion,

$$\frac{3S}{D^3}(C-A)\sin\theta\cos\theta - C\omega\dot{\psi}\sin\theta + A\dot{\psi}^2\sin\theta\cos\theta = A\ddot{\theta}. \quad (1)$$

$$C\omega\dot{\theta} - A\dot{\psi}\cos\theta\dot{\theta} = AD_t(\dot{\psi}\sin\theta), \quad (2)$$

where the symbols have the usual significance. Put

$\frac{3S}{D^3}(C-A) = K$, and multiply (1) by $\dot{\theta}$, and (2) by $\dot{\psi}\sin\theta$, and add, and integrate.

$$K\left(\frac{\sin^2\theta}{2} - \frac{\sin^2\theta_0}{2}\right) = \frac{A\dot{\theta}^2}{2} + \frac{A\dot{\psi}^2\sin^2\theta}{2}. \quad (3)$$

This energy equation states that the work done by the gravitational couple is equal to the kinetic energy imparted. We have also the momental equation, which is

$$C\omega(\cos\theta_0 - \cos\theta) = A\dot{\psi}\sin^2\theta. \quad (4)$$

This is derived from (2) by multiplying it by $\sin\theta$ and integrating, and states that the moment of momentum about OS remains constant.

Substituting the value of $\dot{\psi}\sin\theta$ from (4) in (3),

$$\dot{\theta}^2 = \frac{K}{A}(\cos^2\theta_0 - \cos^2\theta) - \left(\frac{C\omega}{A}\right)^2\left(\frac{\cos\theta_0 - \cos\theta}{\sin\theta}\right)^2$$

$$= (\cos\theta_0 - \cos\theta)\left[\frac{K}{A}(\cos\theta_0 + \cos\theta) - \left(\frac{C\omega}{A}\right)^2\left(\frac{\cos\theta_0 - \cos\theta}{\sin^2\theta}\right)\right].$$

Hence, besides the initial value, $\cos\theta = \cos\theta_0$, $\dot{\theta}$ will become zero when

$$\frac{K}{A}(\cos\theta_0 + \cos\theta) = \left(\frac{C\omega}{A}\right)^2\left(\frac{\cos\theta_0 - \cos\theta}{\sin^2\theta}\right). \tag{5}$$

This is a cubic equation in $\cos\theta$. If ω is large compared with K, it is evident that for a very slight increase of θ, equation (5) will be satisfied, and that for a value beyond this, $\dot{\theta}$ will become imaginary. The axis will, therefore, oscillate between θ_0 and this other limiting value. We thus see that it describes a fluted cone about OS as an axis, in a positive direction. The path is indicated by the left hand side of Fig. 10. If the force is repulsive, instead of attractive, the precession will be negative, or in a retrograde direction, as indicated in the right half of Fig. 10, and the nutations will be inside the precessional circle.

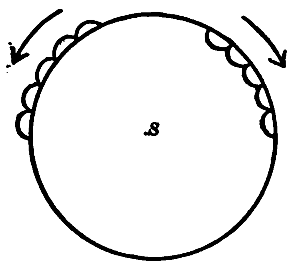

FIG. 10.

If ω is very large, $\dot{\psi}$ and $\dot{\theta}$ become very small, so that we can neglect the squares and products of these small quantities, in comparison with their first powers, without appreciable error. We may also consider $K\sin\theta\cos\theta$ as practically constant during the minute motion. Hence the equations of motion (1), (2), become

$$K\sin\theta\cos\theta - C\,\omega\dot{\psi}\sin\theta = A\ddot{\theta}.$$

$$C\omega\dot{\theta} = AD_t(\dot{\psi}\sin\theta).$$

Taking $\psi \sin\theta$ as x, and θ as y, and the origin of coördinates at the beginning of motion from rest, we have

$$K \sin\theta \cos\theta - C\omega\dot{x} = A\ddot{y}. \tag{6}$$

$$C\omega\dot{y} = A\ddot{x}. \tag{7}$$

Integrating, $$K \sin\theta \cos\theta t - C\omega x = A\dot{y}. \tag{8}$$

$$C\omega y = A\dot{x}. \tag{9}$$

Writing the equations

$$x = \frac{K \sin\theta \cos\theta A}{C^2\omega^2}\left(\frac{C\omega t}{A} - \sin\left(\frac{C\omega t}{A}\right)\right), \tag{10}$$

$$y = \frac{K \sin\theta \cos\theta A}{C^2\omega^2}\left(1 - \cos\left(\frac{C\omega t}{A}\right)\right), \tag{11}$$

we see that they are the integrals of (9) and (8) respectively.

Equations (10) and (11) represent a cycloid having a generating circle with radius $\frac{K \sin\theta \cos\theta A}{C^2\omega^2}$, and the rate of rolling of the circle is $\frac{C\omega}{A}$. The problem is very similar to that in Art. 7. The time of describing a cycloid is $\frac{2\pi A}{C\omega}$. The time of a complete rotation of the body about its axis is $\frac{2\pi}{\omega}$. Hence the ratio of the time of describing a cycloid to a complete rotation is $\frac{A}{C}$. In the case of the earth, where $\frac{C - A}{C} = 0.0032$, the time would be 0.9968 of a sidereal day. The time of the complete precessional period is, under the conditions of the problem, $\frac{2\pi C\omega}{K \cos\theta_0}$. If the earth and sun were the bodies considered in our problem, the nutations would be far too minute to be measured, 0.1″ being

the smallest angle which by the greatest refinements it is possible to measure by astronomical instruments. This angle corresponds to about 12 feet on the surface of the earth.

12. Forced Nutations.

In Fig. 11, the circle *BAVN* represents the orbit of the earth, *ON* the line of nodes, or the intersection of the earth's

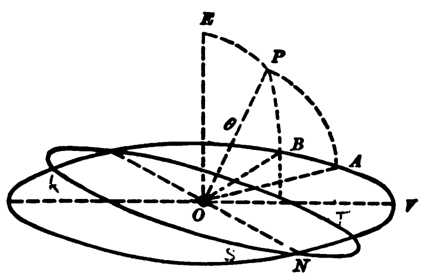

FIG. 11.

equatorial plane with the plane of the ecliptic. *E* is the pole of the ecliptic, *P* that of the equator. The angle *EOP* is the inclination of the earth's axis to the pole of the ecliptic and is θ. The plane *EOB* is thus $\perp$ to the ecliptic and equatorial planes.

In studying the attraction of a body on the earth, it is indifferent whether we consider the earth revolving about the body, or the body revolving about the earth in the same orbit, with the earth's center fixed. We shall suppose the earth's center fixed and the body (sun or moon) revolving in a corresponding orbit about the earth.

Let us suppose the initial position to be at *V* (the vernal equinox). That is, the attracting body is at *V*, and the line of nodes, *ON*, likewise at *V*. Let the angular velocity of the attracting body in its orbit, which is supposed circular, be constant and equal to a. Let the angle made by the line

of nodes with the initial position be $\psi = VON$. This angle ψ is the precession. Let us suppose that at any instant the attracting body is at some point A in its orbit, measured by the angle VOA. Then the angle between the body and the line of nodes is $at - \psi$. The gravitational couple is always about an equatorial axis $\perp$ to the plane AOP, containing the attracting body and the earth's axis, OP. If δ be the declination of the body at any time, then this couple is $K \sin \delta \cos \delta$, where $K = \frac{3S}{D^3}(C - A)$.

We can resolve this couple into a component about an equatorial axis $\perp$ to the plane EOB — the $\dot{\theta}$ component — and a component about an equatorial axis $\perp$ to the former — the $\dot{\psi} \sin \theta$ component or axis.

Since PBA is a right spherical triangle,

$$\cos PA = \cos PB, \cos BA, \text{ or}$$
$$\sin \delta = \sin \theta \sin (at - \psi). \tag{1}$$

Hence the gravitational couple is

$$-K \sin \theta \sin (at - \psi) \sqrt{1 - \sin^2 \theta \sin^2 (at - \psi)}.$$

We use the minus sign because its tendency is to decrease the angle θ. We must resolve this couple into the $\dot{\theta}$ and $\dot{\psi} \sin \theta$ components.

Let A be the angle APB, the angle between the planes AOP and BOP. Then the $\dot{\theta}$ component is the couple into $\cos A$, and the $\dot{\psi} \sin \theta$ component is the couple into $\sin A$. By spherical trigonometry,

$$\tan A = \frac{ctn\,(at - \psi)}{\cos \theta}.$$

Hence

$$\sin A = \frac{ctn\,(at - \psi)}{\sqrt{\cos^2 \theta + ctn^2\,(at - \psi)}}.$$

$$\cos A = \frac{\cos \theta}{\sqrt{\cos^2 \theta + ctn^2\,(at - \psi)}}.$$

The couple about the line of nodes, or the $\dot{\theta}$ component, is thus

$$-K \sin\theta \sin(at-\psi)\sqrt{1-\sin^2\theta\sin^2(at-\psi)} \cdot \frac{\cos\theta}{\sqrt{\cos^2\theta + ctn^2(at-\psi)}}.$$

But

$$\sqrt{\cos^2\theta + ctn^2(at-\psi)} = \frac{\sqrt{1-\sin^2\theta\sin^2(at-\psi)}}{\sin(at-\psi)}.$$

Hence the $\dot{\theta}$ couple is

$$-K\sin\theta\cos\theta\sin^2(at-\psi). \qquad (2)$$

Similarly we find that the $\dot{\psi}\sin\theta$ component is

$$K\sin(at-\psi)\cos(at-\psi)\sin\theta. \qquad (3)$$

We can, therefore, write the equations of motion,

$$-K\sin\theta\cos\theta\sin^2(at-\psi) - C\omega\dot{\psi}\sin\theta + A\dot{\psi}^2\sin\theta\cos\theta = A\ddot{\theta}. \qquad (4)$$

$$K\sin\theta\sin(at-\psi)\cos(at-\psi) + C\omega\dot{\theta} - A\dot{\psi}\cos\theta\dot{\theta} = AD_t(\dot{\psi}\sin\theta). \qquad (5)$$

Multiplying (5) by $\sin\theta$ and integrating,

$$K\int\sin^2\theta\sin(at-\psi)\cos(at-\psi)\,dt = C\omega(\cos\theta - \cos\theta_0) + A\dot{\psi}\sin^2\theta. \qquad (6)$$

This equation states that the time integral of the component of the gravitational couple about the axis OE is equal to the increase (or decrease) of the moment of momentum about that axis — a self-evident fact. As the sign of $\sin(at-\psi)\cos(at-\psi)$ changes with each quadrant, it is clear that the increase of the moment of momentum about OE effected in one quadrant will be undone in the next, and so the average moment of momentum about OE must remain constant.

Multiplying (4) by $\dot{\theta}$ and (5) by $\sin\theta\,(a-\dot{\psi})$,

$$-K\sin^2(at-\psi)\sin\theta\cos\theta\dot{\theta}-\dot{\psi}\sin\theta(C\omega\dot{\theta}-A\dot{\psi}\cos\theta\dot{\theta})=A\dot{\theta}\ddot{\theta}.$$

$$K\sin(at-\psi)\cos(at-\psi)\sin^2\theta\,(a-\dot{\psi})$$
$$-\dot{\psi}\sin\theta\,(C\omega\dot{\theta}-A\dot{\psi}\cos\theta\dot{\theta})$$
$$+a\sin\theta\,(C\omega\dot{\theta}-A\dot{\psi}\cos\theta\dot{\theta})=aA\sin\theta D_t(\dot{\psi}\sin\theta)$$
$$-A\dot{\psi}\sin\theta D_t(\dot{\psi}\sin\theta).$$

Subtracting,

$$-K\sin^2(at-\psi)\sin\theta\cos\theta\dot{\theta}$$
$$-K\sin(at-\psi)\cos(at-\psi)\sin^2\theta\,(a-\dot{\psi}).$$

$$-aC\omega\sin\theta\dot{\theta}+aA\dot{\psi}\sin\theta\cos\theta\dot{\theta}=A\dot{\theta}\ddot{\theta}$$
$$-aA\sin\theta D_t(\dot{\psi}\sin\theta)+A\dot{\psi}\sin\theta D_t(\dot{\psi}\sin\theta).$$

Integrating,

$$-K\int\sin^2(at-\psi)\sin\theta\cos\theta\dot{\theta}$$
$$+K\int\sin(at-\psi)\cos(at-\psi)\sin^2\theta\dot{\psi}.$$
$$-K\int\sin(at-\psi)\cos(at-\psi)\sin^2\theta\cdot adt$$
$$+a\,[C\omega(\cos\theta-\cos\theta_0)+A\dot{\psi}\sin^2\theta]$$
$$=\frac{A\dot{\theta}^2}{2}+\frac{A\dot{\psi}^2\sin^2\theta}{2}=T.$$

Now the first term is the work done by the $\dot{\theta}$ gravitational component, and the second term is the work done by the $\dot{\psi}\sin\theta$ gravitational component, while by (6) the two last terms obliterate each other. Hence the work done by the two gravitational components is equal to the kinetic energy imparted. Performing the actual integration, we have

$$-K\sin^2(at-\psi)\frac{\sin^2\theta}{2}+a\,[C\omega\,(\cos\theta-\cos\theta_0)+A\dot{\psi}\sin^2\theta]=T. \tag{7}$$

Now the first term of (7) is the work done in the plane AOP, for the work done in this plane is $-K\int \sin\delta\cos\delta \cdot d\delta$

$$= -K\frac{\sin^2\delta}{2} = -K\sin^2(at-\psi)\frac{\sin^2\theta}{2}, \quad \text{by (1).}$$

The second term is, by (6), the work done by the gravitational component about OE, through the angle at. Hence the total work done on the body can be decomposed into two parts: that done in the plane, AOP, and that done about the vertical axis, OE. The latter is equal to the increase (or decrease) of the moment of momentum about OE multiplied by the orbital angular velocity. It will be seen that the expression is of the order of work or energy, since it contains angular velocities to the second degree. As before stated, the work done about OE in one quadrant will be undone in the next quadrant. If the attracting body revolves in a retrograde direction, the equation of energy becomes

$$-K\sin^2(at-\psi)\frac{\sin^2\theta}{2} - a[C\omega(\cos\theta - \cos\theta_0) + A\dot{\psi}\sin^2\theta] = T,$$

and the nutations become less than in the former case.

We shall determine the exact nature of the motion. Returning to the equations of motion (4), (5), it is evident that for a great value of ω, $\dot{\psi}$ the precessional velocity, and $\dot{\theta}$ become very small. Hence, without appreciable error, we can neglect squares and products of these small quantities. For a short interval of time, the angle, $at - \psi$, will differ inappreciably from at, since a is very large compared with $\dot{\psi}$. Likewise, the inclination, θ, can be considered as practically constant. Hence our equations become, to a near degree of approximation,

$$-K\sin^2 at \sin\theta\cos\theta - C\omega\dot{\psi}\sin\theta = A\ddot{\theta}. \qquad (8)$$

$$K\sin at \cos at \sin\theta + C\omega\dot{\theta} = AD_t(\dot{\psi}\sin\theta). \qquad (9)$$

We shall now make a short digression into harmonic motion.

Let the ellipse in Fig. 12 have a horizontal semi-axis OA equal to $\frac{K}{4\,aC\omega}\sin\theta\cos\theta$, and a vertical semi-axis $OB = \frac{K}{4\,aC\omega}\sin\theta$. Let us suppose a material point moving in this ellipse with a constant angular velocity $2\,a$, and that the central point exerts an attraction on the point proportional to the distance r and equal to $4\,ra^2$. It will be seen that the motion is stable, for at the points A and B the centrifugal force is exactly equal to the attractive force and opposite in direction. Such a motion is called elliptic harmonic motion. Let us take for coördinates, in the direction OA, $\psi\sin\theta$, and for coördinates in the direction OB, θ. Let us further suppose that the point is moving in a positive direction, or from B to A, and B is the point from which the time is measured. The origin of coördinates is O. Hence we have

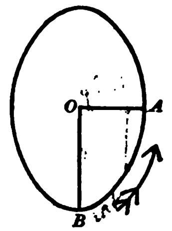

FIG. 12.

$$\left.\begin{aligned}
\psi\sin\theta &= \frac{K}{4\,aC\omega}\sin\theta\cos\theta\sin(2\,at),\\
\dot\psi\sin\theta &= \frac{K}{4\,aC\omega}\sin\theta\cos\theta\cos(2\,at)\,2\,a,\\
\ddot\psi\sin\theta &= -\frac{K}{4\,aC\omega}\sin\theta\cos\theta\sin(2\,at)\,4\,a^2.
\end{aligned}\right\}\tag{10}$$

Likewise,

$$\left.\begin{aligned}
\theta &= \frac{K}{4\,aC\omega}\sin\theta\cos(2\,at),\\
\dot\theta &= -\frac{K}{4\,aC\omega}\sin\theta\sin(2\,at)\,2\,a,\\
\ddot\theta &= -\frac{K}{4\,aC\omega}\sin\theta\cos(2\,at)\,4\,a^2.
\end{aligned}\right\}\tag{11}$$

Let us now suppose that our ellipse is moving bodily to the left, while the point is moving with the harmonic motion already stated. The negative horizontal velocity of the ellipse, as a whole, we shall make $-\frac{K}{2C\omega}\sin\theta\cos\theta$. The compounded horizontal velocity of the point is now

$$\frac{K}{2C\omega}\sin\theta\cos\theta[\cos 2at-1]=-\frac{K}{C\omega}\sin\theta\cos\theta\sin^2 at=\underline{\dot{\psi}\sin\theta}. \tag{12}$$

The vertical angular velocity $\dot{\theta}$ remains unaltered. Let us now substitute for the material point, the end of the earth's axis, and let it move with the same velocity in this moving ellipse. If the earth were not rotating, we should have to constrain the end of the axis by material restraints and apply outside forces to make it move in the ellipse as we desire. If it is rotating with angular velocity ω, gyroscopic couples will be set up. The gravitational couple about the $\dot{\theta}$ axis is $-K\sin^2 at\sin\theta\cos\theta$, and that about the $\dot{\psi}\sin\theta$ axis is $K\sin at\cos at\sin\theta$.

The gyroscopic couple about the $\dot{\theta}$ axis, due to the angular velocity $\underline{\dot{\psi}\sin\theta}$, will be $-C\omega\,\underline{\dot{\psi}\sin\theta}$, and the gyroscopic couple about the $\dot{\psi}\sin\theta$ axis will be $C\omega\dot{\theta}$. But from equations (11) and (12), these couples are $K\sin^2 at\sin\theta\cos\theta$ and $-K\sin\theta\frac{\sin 2at}{2}=-K\sin at\cos at\sin\theta$. These are exactly equal and opposite to the gravitational couples. Hence the axis will need no restraints, but will move of itself in the path we have prescribed. We have thus reduced a dynamical to a simple kinematical problem. It will be seen that by equating the gravitational and gyroscopic forces and integrating, we shall obtain the equations of the curve traced. The motion of the axis, therefore, under the attraction of a revolving (performing a revolution)

body, is a harmonic motion in a positive direction in an ellipse having a major semi-axis $\frac{K}{4\,aC\omega}\sin\theta$, and a minor semi-axis $\frac{K}{4\,aC\omega}\sin\theta\cos\theta$. It moves in this ellipse harmonically with a constant angular velocity $2\,a$, while the ellipse itself moves with the constant horizontal angular velocity $-\frac{K}{2\,C\omega}\sin\theta\cos\theta$. The long axis of the ellipse always points towards the pole of the orbit. The constant retrograde precessional velocity of the center of the ellipse is $-\frac{K}{2\,C\omega}\cos\theta_0$, where θ_0 is its constant inclination to the pole, or $\dot{\psi} = -\frac{K}{2\,C\omega}\cos\theta_0$.

The constant, or regular, precession due to the sun is about 15″ a year: that due to the moon is nearly $2\frac{1}{2}$ greater, so that the regular annual precession of the earth's axis is about 50″.

The semi-axis of the ellipse OB is for the sun about 0.53″, while the semi-axis OA is about 0.48″. The axes of the ellipse due to the moon are about $\frac{1}{6}$ as large, or barely within the range of measurability.*

The ellipse always maintains its position with respect to the line of nodes. This is necessarily so, as it is a forced nutation. When the attracting body is at a node (vernal or autumnal equinox), the axis is at the lowest point of the ellipse and at rest, for here the constant precessional

* In computing K, we must, of course, use astronomical units. $\frac{C-A}{A} = .0032$. For the sun, the attraction $\frac{SM}{D^2}$ must equal the centrifugal force of the earth, which is Ma^2D, M being the earth's mass, and a her orbital angular velocity.

Hence $$\frac{S}{D^3} = a^2.$$

velocity of the ellipse as a whole is exactly equal and opposite to the velocity of the axis *in* the ellipse (see Equations (10) and (11)). If, therefore, we started our rotating body with its axis at rest, it would immediately fall into motion in that part of the harmonic ellipse necessary to bring it to rest at the node. It is for this reason that the axis of the moon always lies in a plane $\perp$ to the intersection of her orbit with the ecliptic, or, stated otherwise, the plane of the moon's orbit, the plane of the moon's equator, and the plane through the center of the moon $||$ to the ecliptic always intersect in the same line. For, we can consider the moon's center to be fixed, and the sun to revolve about it in its corresponding orbit in the plane of the ecliptic, while the earth revolves about it in the plane of the moon's orbit. Let us suppose that the moon's equatorial plane intersects these two orbital planes in two different lines. The forced nutations we have just considered will strive to set the moon's axis $\perp$ to each of these lines, and this can only be accomplished when the two lines of nodes coincide. The result is that the moon's axis moves as if it were rigidly attached to the plane of her orbit. This is known as Cassini's theorem, and was discovered by Cassini from observation.

The extreme range of variation of the inclination of the earth's axis, from these forced nutations, is about $1.1''$, while the extreme range of variation of celestial longitude from the mean is about $1''$. Since the velocity in the ellipse is $2\,a$, while the orbital velocity is a, it will be seen that two complete ellipses are described in every complete revolution to the same node. The axis is nearest to the pole of the ecliptic at the solstices, while it is farthest away at the equinoxes. The axis would make a complete precessional circle in $\frac{2\pi}{50''}$, = 26,000 years, about.

For a single attracting body, the path of the axis is like that represented on the right side of Fig. 10. Two of these curves are described during every orbital period, while in the free nutations one of the cycloids is described in a period somewhat less than the rotational period. The actual forced nutations of the earth are relatively large, while if there were free nutations, which, however, do not exist, they would be extremely minute.

13. Gyroscopic Motion of the Moon's Orbit.

The plane of the moon's orbit makes an angle of nearly 5° with the plane of the ecliptic. If the mass of the moon were uniformly distributed over her orbit into a thin ring of matter, and this ring were rotating in its plane, with the same angular velocity as the moon in her orbit, the attraction of the sun upon this ring would set up gyroscopic couples. Now the moon, by her motion, practically distributes her mass over her orbit, so that, the motion being rather rapid, the attraction of the sun is practically the same as if it were acting upon such a rotating ring. It is, therefore, a simple matter to calculate the motion of such a ring. C, the maximum moment of inertia, would be MD^2, where M is the mass of the moon and D its distance from the earth. $A = \dfrac{MD^2}{2}$. It is clear that the rotational axis of this ring, i.e., the pole of the moon's orbit, will describe, in a retrograde direction, a small precessional circle about the pole of the ecliptic, at a distance of 5°. It can easily be calculated that the time of describing one complete precessional circle will be about $18\frac{2}{3}$ years. The rapidity of the precession, compared with that of the earth, is due to the very much greater moment of momentum of the ring about its axis, and to the greater gravitational couple. It is further clear that the motion of the ring will

be precisely like that of the earth's axis, just discussed in the previous article. The axis will move in an ellipse with harmonic motion, in a positive direction, and the major axis of the ellipse will always be directed towards the pole of the ecliptic. The center of the ellipse will move with a constant retrograde precession about the pole of the ecliptic. The motion in the ellipse is due to the forced nutations arising from the $\dot{\psi} \sin \theta$ and $\dot{\theta}$ components of the gravitational couple. These forced nutations, it appears, have not been considered in the theory of the moon's motion: at least, the author has been unable to find such corrections in lunar tables. In any case, they must be applied, and it is believed that they may in part, at least, if not wholly, explain certain anomalies in her motion. At present, the position of the moon, as given by the best tables, may be "out" $3''$ or $4''$, corresponding to 3 or 4 miles in her orbit. These deviations are now on one side, now on the other, of her calculated position. Two of these ellipses will be described in every synodical period.

14. Effect of the Earth's Equatorial Protuberance on the Motion of the Moon's Orbital Plane.

Regarding the moon as a uniform ring of matter rotating about the earth, it will be seen that, as the plane of this ring and the plane of the earth's equator are not coincident, but are inclined at an angle varying from 18° to 28°, the earth's equatorial mass will set up a gyroscopic motion in this ring, precisely similar to the motions we have just considered. It will be very slight, however. Taking the average inclination as 23°, the complete precessional period would be about 159,000 years. This is at its present rate, but the factors are continually changing, so that long before this period is ended, the rate will have changed. This

precession, of course, takes place about the pole of the earth and is retrograde.

15. Effect of the Moon's Orbital Precession upon the Axis of the Earth.

We have seen that the pole of the moon's orbit moves about the pole of the ecliptic in a retrograde direction in about $18\frac{2}{3}$ years. Let us examine what effect this has upon the motion of the earth's axis. We have seen that a revolving bi-axial body executes, under the influence of the attracting body, a steady retrograde precessional movement, on which are superposed certain forced nutations. We have also seen that, if the attracting body is placed at the pole of the orbit and supposed to exert a repulsional instead of an attractional force, we shall have the same steady retrograde precession, on which are superposed certain free nutations. Disregarding the nutations for the time being, the steady precessional motion will be the same in either case, if we choose a suitable mass for the repelling body. We shall, therefore, regard the center of the earth fixed, and the moon placed at the pole of her orbit, at B in Fig. 13, and moving in a retrograde direction about the pole of the ecliptic C. Let A be the position of the earth's axis, and the angle $CA = \theta$, its inclination to the pole of the ecliptic. The angle $CB = a$ is actually about $5°$. The angle $BA = c$ is, therefore, the inclination of the earth's axis to the repelling body. Let the angular velocity of the point B in the small circle be $-b$, and the angle CAB be designated by A, while the angle ACB is designated by C. The gravitational couple is, therefore, $-K \sin c \cos c$. We can resolve this into the two component couples, $-K \sin c \cos c \cos A$, which is the $\dot{\theta}$ couple, and $K \sin c \cos c \sin A$, which is the $\dot{\psi} \sin \theta$ couple. We are

careful to write the signs thus, because the first always tends to decrease the value of θ, while the second, the $\dot{\psi} \sin\theta$ component, acts in a positive or negative direction according to the sign of $\sin A$. In Fig. 13, A is a negative angle and in this position the couple will act in a negative direction, while, when B is on the other side of CA, it will act in a positive direction. Taking a fixed vertical plane

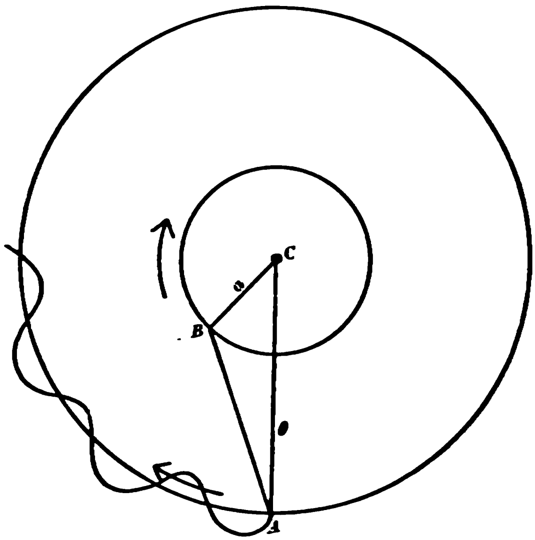

Fig. 13.

through C as the standard of reference, the angle $ACB = C$ will be $(-bt - \psi)$, where, as before, ψ is the precession of A about the pole of the ecliptic.

By spherical trigonometry, $$\frac{\sin A}{\sin C} = \frac{\sin a}{\sin c}, \tag{1}$$

and $$\cos c = \cos a \cos\theta + \sin a \sin\theta \cos C. \tag{2}$$

Hence the $\dot{\theta}$ couple is $-K \cos c \sqrt{\sin^2 c - \sin^2 a \sin^2 C}$.

Now, since $\sin^2 c - \sin^2 a \sin^2 C =$

$$1 - [\cos^2 a \cos^2 \theta + 2 \sin a \cos a \sin \theta \cos \theta \cos C + \sin^2 a \sin^2 \theta \cos^2 C] - \sin^2 a \sin^2 C =$$

$$1 - \cos^2 a \cos^2 \theta - 2 \sin a \cos a \sin \theta \cos \theta \cos C - \sin^2 a[\cos^2 C + \sin^2 C] + \sin^2 a \cos^2 \theta \cos^2 C$$

$$= \cos^2 a \sin^2 \theta - 2 \sin a \cos a \sin \theta \cos \theta \cos C + \sin^2 a \cos^2 \theta \cos^2 C$$

$$= (\cos a \sin \theta - \sin a \cos \theta \cos C)^2,$$

therefore

$$\sqrt{\sin^2 c - \sin^2 a \sin^2 C} = \cos a \sin \theta - \sin a \cos \theta \cos C.$$

The $\dot{\theta}$ couple is, then,

$$- K \cos c \, (\cos a \sin \theta - \sin a \cos \theta \cos C).$$

Substituting the value of $\cos c$ from (2), this is

$$- K(\cos a \cos \theta + \sin a \sin \theta \cos C)(\cos a \sin \theta - \sin a \cos \theta \cos C)$$

$$= - K[\cos^2 a \sin \theta \cos \theta - \sin a \cos a \cos^2 \theta \cos C + \sin a \cos a \sin^2 \theta \cos C - \sin^2 a \sin \theta \cos \theta \cos^2 C]$$

$$= - K \cos^2 a \sin \theta \cos \theta + K \sin a \cos a \cos C \cos 2\theta + K \sin^2 a \sin \theta \cos \theta \cos^2 C. \quad (3)$$

The $\dot{\psi} \sin \theta$ couple is

$$K \sin c \cos c \sin A = K \cos c \sin a \sin C$$
$$= K \sin a \sin C (\cos a \cos \theta + \sin a \sin \theta \cos C)$$
$$= K \sin a \cos a \sin C \cos \theta + K \sin^2 a \sin C \cos C \sin \theta. \quad (4)$$

Our equations of motions are, therefore,

$$K\sin^2 a\cos^2(-bt-\psi)\sin\theta\cos\theta$$
$$+K\sin a\cos a\cos(-bt-\psi)\cos 2\theta$$
$$-K\cos^2 a\sin\theta\cos\theta - C\omega\dot\psi\sin\theta + A\dot\psi^2\sin\theta\cos\theta = A\ddot\theta \quad (5)$$

$$K\sin^2 a\sin(-bt-\psi)\cos(-bt-\psi)\sin\theta$$
$$+K\sin a\cos a\sin(-bt-\psi)\cos\theta$$
$$+C\omega\dot\theta - A\dot\psi\cos\theta\dot\theta = AD_t(\dot\psi\sin\theta). \quad (6)$$

[*Note.* We have previously used C to denote an angle, but now it has its usual significance. There will be no confusion.]

Multiply (6) by $\sin\theta(-b-\dot\psi)$,

$$K\sin^2 a\sin(-bt-\psi)\cos(-bt-\psi)\sin^2\theta(-b-\dot\psi)$$
$$+K\sin a\cos a\sin(-bt-\psi)\sin\theta\cos\theta(-b-\dot\psi)$$
$$+C\omega\sin\theta\dot\theta(-b-\dot\psi) - A\dot\psi\sin\theta\cos\theta(-b-\dot\psi)\dot\theta$$
$$=A(-b-\dot\psi)\sin\theta D_t(\dot\psi\sin\theta). \quad (7)$$

Integrating (5),

$$K\sin^2 a\cos^2(-bt-\psi)\frac{\sin^2\theta}{2}$$
$$+K\sin^2 a\int\sin^2\theta\sin(-bt-\psi)\cos(-bt-\psi)(-b-\dot\psi)$$
$$+K\sin a\cos a\sin\theta\cos\theta\cos(-bt-\psi)$$
$$+K\sin a\cos a\int\sin\theta\cos\theta\sin(-bt-\psi)(-b-\dot\psi)$$
$$-K\cos^2 a\frac{\sin^2\theta}{2} - C\omega\int\dot\psi\sin\theta\dot\theta + A\int\dot\psi^2\sin\theta\cos\theta\dot\theta = A\frac{\dot\theta^2}{2}. \quad (8)$$

Integrating (7),

$$K\sin^2 a\int\sin(-bt-\psi)\cos(-bt-\psi)\sin^2\theta(-b-\dot\psi)$$
$$+K\sin a\cos a\int\sin\theta\cos\theta\sin(-bt-\psi)(-b-\dot\psi)$$
$$+bC\omega\cos\theta - C\omega\int\dot\psi\sin\theta\dot\theta + bA\int\dot\psi\sin\theta\cos\theta\dot\theta$$

$$+A\int\dot{\psi}^2\sin\theta\cos\theta\dot{\theta} = -bA\dot{\psi}\sin^2\theta + bA\int\dot{\psi}\sin\theta\cos\theta\dot{\theta}$$
$$-\frac{A\dot{\psi}^2\sin^2\theta}{2}. \qquad (9)$$

Subtracting (9) from (8),

$$K\sin^2 a\cos^2(bt+\psi)\frac{\sin^2\theta}{2} + K\sin a\cos a\sin\theta\cos\theta\cos(bt+\psi)$$
$$-K\cos^2 a\frac{\sin^2\theta}{2} - b\,[C\omega\cos\theta + A\dot{\psi}\sin^2\theta] = T + \text{Const.}$$

$$\text{Const.} = K\sin^2 a\frac{\sin^2\theta_0}{2} + K\sin a\cos a\sin\theta_0\cos\theta_0$$
$$-K\cos^2 a\frac{\sin^2\theta_0}{2} - bC\omega\cos\theta_0.$$

Hence we have the equation of energy,

$$K\sin^2 a\cos^2(bt+\psi)\frac{\sin^2\theta}{2} - K\sin^2 a\frac{\sin^2\theta_0}{2}$$
$$+K\sin a\cos a\sin\theta\cos\theta\cos(bt+\psi)$$
$$-K\sin a\cos a\sin\theta_0\cos\theta_0 - K\cos^2 a\frac{\sin^2\theta}{2} + K\cos^2 a\frac{\sin\theta_0}{2}$$
$$-b\,[C\omega(\cos\theta-\cos\theta_0) + A\dot{\psi}\sin^2\theta] = T. \qquad (10)$$

The terms not in the bracket express the work done in the plane BA, for this work is

$$-K\int\sin c\cos c\,dc = K\left(\frac{\sin^2 c_0}{2} - \frac{\sin^2 c}{2}\right), \quad \text{and}$$

$$\sin^2 c = \sin^2 a + \cos^2 a\sin^2\theta - 2\sin a\cos a\sin\theta\cos\theta\cos C$$
$$-\sin^2 a\sin^2\theta\cos^2 C. \quad \text{Hence} \quad K\left(\frac{\sin^2 c_0}{2} - \frac{\sin^2 c}{2}\right)$$
$$= K\sin^2 a\frac{\sin^2\theta}{2}\cos^2 C - K\sin^2 a\frac{\sin^2\theta_0}{2}$$
$$+K\sin a\cos a\sin\theta\cos\theta\cos C$$
$$-K\sin a\cos a\sin\theta_0\cos\theta_0 + K\frac{\cos^2 a}{2}(\sin^2\theta_0 - \sin^2\theta).$$

As in Art. 12, the bracket expresses the work done about an axis $\perp$ to the plane of the ecliptic, and is equal to the increase (or decrease) of the moment of momentum about this axis, multiplied by the precessional velocity of the moon's orbit. The average of this work for a complete motion of the repelling body around the small circle, from conjunction to conjunction, is zero.

When ω is large, as in the case of the earth, $\dot{\psi}$ and $\dot{\theta}$ must be very small, and we shall neglect their squares and products. $bt + \psi$ will be very nearly bt for a short time, and θ will remain practically constant. Hence we can use as the equations of motion to a close degree of approximation, putting $\dot{\psi} \sin\theta = x$, $\dot{\theta} = \dot{y}$,

$$K \sin^2 a \sin\theta \cos\theta \cos^2 bt + K \sin a \cos a \cos 2\theta \cos bt - K \cos^2 a \sin\theta \cos\theta - C\omega\dot{x} = A\ddot{y} \qquad (11)$$

$$- K \sin^2 a \sin\theta \sin bt \cos bt - K \sin a \cos a \cos\theta \sin bt + C\omega\dot{y} = A\ddot{x}. \qquad (12)$$

Now if we use only the first terms of the left members of Equations (11) and (12), we have precisely the case treated in Art. 12, and the motion due to these first terms is a harmonic motion in an ellipse with the ellipse itself moving with a constant retrograde precessional velocity. Taking the second terms only of (11) and (12), it is evident that the motion due to them will be simply an elliptic harmonic motion in an ellipse having axes $K \sin a \cos a \cos 2\theta$ and $K \sin a \cos a \cos\theta$. The motion in this ellipse will be retrograde and the constant angular velocity $- b$, while in the other ellipse the velocity is $2b$. If we take the third terms of (11) and (12), which are respectively $- K \cos^2 a \sin\theta \cos\theta$ and zero, the result will be a constant retrograde precessional velocity. The actual motion of the pole of the earth will, therefore, be the resultant or these three separate motions. It is clear that there will be a continual, though

varying, precession in a retrograde direction, while the inclination to the pole of the ecliptic will pass through periodically-recurring values. Integrating (12),

$$C\omega y - A\dot{x} = K \sin^2 a \sin\theta \frac{\sin^2 bt}{2\,b}$$
$$- K \sin a \cos a \cos\theta \left(\frac{\cos bt - 1}{b}\right). \qquad (13)$$

Since ω is very large compared to $\dot{x}$, we can neglect the latter. Hence starting with the moon on the same celestial meridian, the axis of the earth will at first fall away from the pole of the ecliptic, reaching a maximum when $bt = \pi$. After this it will fall in, reaching a minimum again when $bt = 2\,\pi$, and so on, alternately falling outward and inward from the circle of mean inclination. One complete wave will be executed in every complete circuit, bringing the moon and pole of the earth to the same meridian. The pole will thus describe a symmetrical wavy line through the circle of mean inclination, as shown in Fig. 13. The integrals of (11) and (12) enable us to plot the curve completely.

The amount of this forced nutation is, comparatively speaking, considerable. That is, it is much greater than the other forced nutations we have considered, and amounts to $9''.2$ for θ. It was the first nutation actually observed, and was discovered by Bradley. In his time, astronomical instruments were not perfect enough to detect the other nutations.

16. The Chandlerian Period.

We have seen that a bi-axial body executing a Poinsot motion performs a constant precession about an invariable line. The angular velocity of this precession is $\dot{\psi} = \dfrac{C\omega}{A \cos y}$,

where y is the inclination of the axis to the invariable line. The instantaneous axis executes a small circle about the axis of the body, and *relatively* to the body moves in a positive direction with the angular velocity $\dot{\psi}\cos y - \omega$. (See Art. 5.)

Since $\dot{\psi}\cos y = \dfrac{C\omega}{A}$, this velocity, which is relative to some fixed point on the body, is $\omega\left(\dfrac{C-A}{A}\right)$. With a high value of ω, the instantaneous axis makes a very small angle with the axis.

Now if we supposed the earth to be executing a Poinsot motion about some invariable line (which it must be distinctly remembered it does not do), since $\dfrac{C-A}{A} = 0.0032+$, the instantaneous axis would make a complete circuit, relatively to some fixed point on the earth's surface, in $\dfrac{A}{C-A}$ sidereal days. This is 312 sidereal days or about 10 months. Hence there would be a ten-monthly period of variation of latitude, as determined by celestial objects, with a maximum difference of 2ι, where ι is the constant angle between the instantaneous axis and the axis of maximum moment, i.e., the axis of the earth. Places 180° apart in longitude would experience opposite variations, a maximum in one place corresponding to a minimum in the other.

This imaginary variation, on the supposition that the earth executed a Poinsot motion about some invariable line, has been called the Eulerian nutation, and its period has been called the Eulerian ten-monthly period. From the previous discussion we have seen that the earth does not execute any Poinsot movements, and the consideration of what variations of latitude would result in case it should execute a Poinsot motion is purely academic. There is no

such thing as an Eulerian nutation, or an Eulerian ten-monthly period.*

It has, however, been found by observation that there are actually small periodic variations of latitude, and that places on opposite meridians experience opposite variations. Chandler discovered a 14-monthly period in these variations, but there is absolutely no trace of a 10-monthly period. The variations are extremely small with maximum differences of about half a second. It is a coincidence, perhaps more, that the periods of the forced solar and lunar nutations have a least common multiple of 14 months.

Meteorological phenomena are, in all probability, responsible for a part of these changes. A cyclone always rotates in the same direction as the surface of the earth under it. The tendency of these great rotating masses of air would be to shift the rotation axis slightly to their side. They occur alternately in the northern and southern hemispheres every six months. The heaviest are on opposite sides of the world, viz., the north Atlantic and the south Pacific. These might give a yearly period in the shift of the rotation axis, and, in fact, traces of a yearly period have been discovered, but the most distinct period seems to be the 14-monthly one. The exact determination of the causes of these changes awaits solution. Among possible factors are the forced nutations of the sun and moon, and the elasticity of the earth.

* The impact of some external body, or a shifting of the principal axes by a geological convulsion, would produce a slight Poinsot motion, but this would quickly be obliterated by the forced nutations.

PART II.

APPLICATIONS.

PART II.

17. Applications.

Children were the first to apply the gyroscope. They did this in their toys, the rolling hoop and the top. Some years ago they used to play with a form of gyroscope called Diablo. The fascination here is that these inanimate objects contradict their every-day experience and seem to be alive. And, in a certain sense, they are alive, for it is probable that all life consists only of more complicated forms of motion. The hoop and the top, when set in rotation, refuse to fall, to the child's great delight. As soon as they begin to fall and acquire the slightest velocity about a horizontal axis, immediately a gyroscopic couple is set up about the $\dot{\psi} \sin \theta$ axis and the resulting velocity about this axis counterbalances the gravitational effort. A hoop, if inclining a little to the left, does not fall but simply turns to the left. If inclining to the right, it turns to the right and continues rolling. If the rotational velocity is high, the inclinations and the corresponding turns become very small, and the hoop runs practically upright in a straight course. This principle, as we shall see, has been applied by Howell to the torpedo.

It is the same with a bicycle. The front wheel turns easily and automatically. It is a rolling hoop — a gyroscope. With a high speed, the bicyclist instinctively feels himself in equilibrium. If there is the slightest tendency to tip to one side or the other, he learns that a touch, the merest trace of a movement about a vertical axis — through the handle bar — sets up a powerful gyroscopic couple

about a horizontal axis which immediately rights him. Of course, if he executes an appreciable curve, centrifugal forces, tending to throw his body to the outside of the curve, act in the same direction, but in direct running the front wheel does the righting. If anybody doubts the force of this gyroscopic couple, let him spin a bicycle wheel on a bar through the axle held in his hands, and give the bar the slightest turn. He will experience a very violent jerk, but not in the direction he expected it. If he gives it too sudden a turn, it will take him off his guard and twist out of his hands.

A top, if it has a sharp point and the supporting surface is rough, will precess about the vertical at a practically constant height. It actually does fall rapidly back and forth through its nutations, but these cannot be seen; only heard as a humming. If, however, the point is rounded, so that it has some surface, and the surface on which it spins is rough, the toe will roll on the surface, thus causing, at the expense of the initial rotational energy, a new rotation about a vertical axis which, as we have seen, will set up a gyroscopic couple tending to bring the axis of the top into coincidence with this vertical axis. And, if the rotational energy is sufficient, it will, in fact, raise the top to an upright position. The top now seems motionless, and it is soundless. No more precession, no more nutations, no more humming. The boys say it is asleep. Nutation means nodding, so that after so much nodding it is eminently proper that it should go to sleep. It would remain asleep forever, if it were not for the friction of the air and that on the point of support.

Whenever vehicles running on wheels turn, gyroscopic couples are set up. When a train rounds a curve, gyroscopic couples are set up tending to overturn the cars outward. It is well known that the simple centrifugal

moment tends to overturn the cars outward and this is provided against by raising the outer rail. Engineers, however, overlook the fact that besides this simple centrifugal moment, there is an *added* gyroscopic couple, due to the rotation of the wheels, which always acts *with* the centrifugal moment, and which helps the centrifugal couple to overturn the car, in case such an accident occurs. It is true that, owing to the mass of the car being much greater than the mass of the wheels, and its center of gravity much higher above the rails, the simple centrifugal moment is much greater than the gyroscopic couple. The gyroscopic couple, however, is very appreciable, and where the limit has been calculated solely upon the basis of the centrifugal couple, as is usually the case, and this limit is approached, the unsuspected gyroscopic couple is the agent which gives the "coup de grace." Such accidents have occurred, but neither before nor afterwards does there seem to have been any clear comprehension of the principles involved. The cloudy ideas enshrouding gyroscopics in the popular mind, and even in that of engineers, is perhaps due on the one side to the fear of certain pure mathematicians * that their work and demonstrations may by any possible means be put to a practical use, and on the other side to the dread and distrust of the practical man of what he styles the higher mathematics.

When an aviator spirals downward in his machine, the aeroplane as a whole becomes a gyroscope, even though the motor be stopped. In general, the axis about which it is turning at any instant is not a principal axis, and when such a rotating body is turned sharply it will be subjected to gyroscopic couples which are especially dangerous, since they act in directions unapprehended and unforeseen by

* Was it not Dirichlet who boasted that nothing of his work could be put to practical use?

the aviator. Where there is a single propeller and motor, the very appreciable gyroscopic action of these two latter bodies is added to that of the aeroplane, when they are rotating. It is a simple matter of calculation to determine how much and in what direction the gyroscopic action is, under given conditions. While without doubt numbers of accidents have been due to other causes, there is equally no doubt that a considerable number of the many fatalities have been due to a lack of knowledge, which is general, and a lack of appreciation of gyroscopic laws. Recently a controversy has raged, with an unnecessary degree of asperity, between the pros and cons of the gyroscopic "*theory*" of aeroplane accidents. Some aeroplanists have denied the existence of gyroscopic action, while others have maintained that even if it exists, it is inappreciable. Unfortunately both sides have had only a hazy idea of the question involved. At a time when hardly a day passes without some fatality being reported, it would seem that the practice of the art should be stopped until the ground has been gone over again and every dynamical principle involved thoroughly investigated and tested.

The axis of C of an aeroplane, i.e., its axis of greatest moment, is generally its longitudinal axis, while its axis of A, or least axis, is along the planes. It is a tri-axial body, and turning freely and rapidly in all directions in the air is subjected to gyroscopic actions which cannot occur in a vehicle moving on the surface of the earth, whether on land or water. Its rotation, or turning, about an axis is wholly independent of its translational motion. Where the motion is wholly translational and the axes maintain parallelism, no gyroscopic action can occur. Where the turning is about a principal axis, the only gyroscopic couple will be that due to the motor and propeller. Whenever an accident occurs during a turn, lateral or vertical, gyroscopic

action should suggest itself. With two motors and propellers turning in opposite directions, gyroscopic action from this source is eliminated, and it would be much safer if this arrangement were adopted on all aeroplanes. Turns should never be made sharply, if it can be avoided, and placing the center of gravity as far as possible below the supporting surface makes for stability. The tail should be a good distance in the rear and have a broad surface.

18. Griffin Grinding Mill.

One of the earliest applications of the gyroscope was the Griffin grinding mill. A heavy steel drum is suspended by a universal joint above the center of a mortar, and hangs below the rim. If it is set in rapid rotation and brought against the rim, it begins to roll, and the gyroscopic couple set up by the change of plane of the rotating drum presses it with great force outward against the rim. It not only holds the drum on the rim against gravity, but exerts a very great pressure. The gyroscopic couple will be readily seen to act in a vertical plane through the center outward. In all these cases the reader can at once determine the axis and direction of the couple by remembering that the rotation and turning axes always strive to coalesce. When a gyroscope rolls along a surface, if some point in the axis is fixed, it clings to the surface like a magnet, following the most intricate curves and even turning about an abrupt edge of 180°. This principle of forcing a gyroscope to precess by using its own rotation to make it roll along a guiding surface, and thereby automatically obtaining a pressure from the precession couple, has been applied by Brennan in his gyro-monorail.

19. Howell Torpedo.

In the Howell torpedo, we have an application of the gyroscopic principle to keeping a body moving in a straight line, similar to the case of a child's hoop. We saw that in the hoop a high rotation reduced the tipping and turning right or left to a minimum. In the hypothetical case of an infinite rotation, a motion of the plane of the gyroscope is impossible under any finite force, and we approach this hypothetical case, *sed longo intervallo*, as we increase the rotation. The Howell torpedo has amidships a heavy flywheel mounted on a horizontal axis $\perp$ to its length. It is spun up by outside power to 10,000 turns per minute. It is this stored rotational energy, and here is the weak point of the torpedo, which is used, not only to guide it on an approximately straight course, but also to furnish the propulsive power. After being launched, the axle automatically gears into two propeller shafts which turn in opposite directions.

Double propellers turning in opposite directions effect two things. In the first place they eliminate all gyroscopic action *per se*. Ships with single propellers, or with paddle wheels, or aeroplanes with a single rotating motor, all set up gyroscopic couples when turning. In the second place, the lowermost blade of a propeller gets a stronger grip on the water, because the pressure is greater there, than does the uppermost blade. This stronger thrust of the lower blade tends to turn the bow of the ship and has to be counteracted by "helm."

If now any couple tends to turn the torpedo about a vertical axis — make it deviate from its course — the motion in this direction would be a minimum, and the action of the couple would be mainly to make the torpedo roll, or turn about its long axis. The torpedo is ballasted

so as to keep, ordinarily, on even keel. After the couple has ceased acting the backward roll will tend to bring it approximately on its course again. In firing, due regard must be had for currents, as, of course, the torpedo moves as readily || to itself as if it had no gyroscope. This torpedo is not now used.

20. The Obry Device.

The Obry device used in the Whitehead torpedo is as follows. A small flywheel rotates with a high velocity about an axis || to the length of the torpedo. The ring holding its axle turns about a horizontal axis in an outer ring, and this outer ring turns about a vertical axis which is fixed to the torpedo. On the outer ring is a pin which engages the forked end of a lever, Fig. 14.

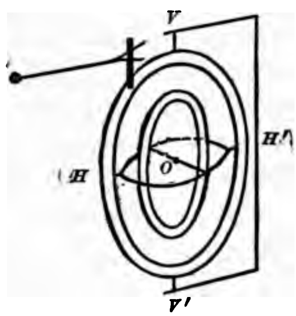

FIG. 14.

If the torpedo turns sideways, since friction on the axes is slight, the flywheel practically maintains its place until one side or the other of the fork is brought against the pin. The action of the fork on the gyroscope is to move it slightly about a vertical axis, but owing to the high velocity of the disc, the resulting gyroscopic couple turns the flywheel about the horizontal axes, and this reacts against the fork. In other words, the lever has to act against what we have called a high rotational inertia. Since the lever is easily moveable, it turns the outer ring only slightly, the inner ring considerably more, while it is itself freely turned by the reaction. It opens an air valve by which compressed air turns two horizontal rudders, thus bringing the torpedo back to its course. If the torpedo deviates to the other

side, the lever is moved in the opposite direction, and the rudders work so as to oppose this change. The range of the Whitehead torpedo has been increased to 4 or 5 miles. The motive power is compressed air. It would seem that storage batteries, even of the present excessive weight, might be used in a large torpedo, both for propulsion and driving the flywheel. Electrical contact with the gyroscope pin would be an ideal method of actuating the rudders.

21. The Schlick Stabilisator.

This method of steadying ships against rolling was devised by Otto Schlick of Hamburg.

A heavy flywheel — 5 tons or more — rotating 1800 turns per minute, 1.6 meters in diameter, is set with its axis vertical in a frame which can turn on trunnions about a horizontal axis athwartships, in the most central part of the ship available. The center of gravity is below the horizontal axis, so that it hangs normally like a pendulum. On the left, Fig. 15, is shown a ring to which is applied a

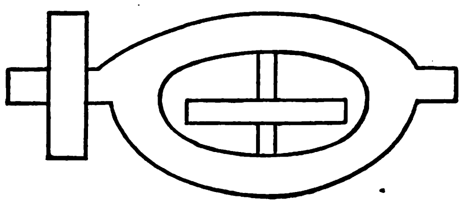

FIG. 15.

brake band, worked either automatically or by hand. In practice it is generally necessary to brake down considerably. The flywheel is driven by steam and is a turbine, the steam entering through one trunnion and escaping by the other. It will be seen that the rolling of the ship causes the flywheel to turn about a fore-and-aft axis. In doing this a gyroscopic couple is set up which turns the flywheel

about its horizontal axis, thus producing a powerful couple acting against the roll. The roll will be very limited while the swing of the pendulum will be considerable. In other words, the swing of the pendulum is substituted in large part for the roll of the ship.

There is much to recommend this device for certain special services and conditions. The objections are its expense and the considerable amount of space and freight which it displaces. A flywheel of 5 tons weight, rotating 1800 times a minute, is a store of energy which might be set loose in an undesired direction and thus possesses potential danger. The couple, acting on the bearings against a roll, might easily be 10 tons on each bearing in a moderate sea way. This stress is communicated to the frames and sides of the ship through two points. Up to a certain size of ship, these strains could be readily borne, while above that size the sides would be liable to buckle at every considerable wave coming abeam. Of course, it would be possible to distribute the strain by having several gyroscopes. In heavy weather, it would probably not be advisable to have the gyroscope running. If a ship were pitching heavily and the gyroscope became jammed, thus pitching *with* the ship, it would roll the ship over dangerously and might capsize it. The outward swing would always have to be kept in phase with the wave running. It would not do, while in the trough of the sea, to have the gyroscope coming back in the wrong direction.

While the size of ship to which the stabilisator is applicable seems to have a very definite limit, it is nevertheless adaptable in certain types such as channel boats, small passenger steamers, torpedo boats and destroyers. It has already demonstrated its utility practically in a number of ships, and its adoption will probably increase with time.

The principle of the Griffin Mill, viz., that of producing

an artificial automatic precession by making the axle engage and roll by its own rotation along a guiding surface, can easily be applied to the stabilization of a ship. Brennan has used this principle in the stabilization of his gyro-monorail, as already pointed out, and it can be used in precisely the same way for the stabilization of a ship. This has been done on a U. S. destroyer.

22. The Brennan Gyro-Monorail.

The Brennan device consists of two such flywheels as that depicted in Fig. 16. The axles of both wheels are normally horizontal and ⊥ to the length of the car, and in

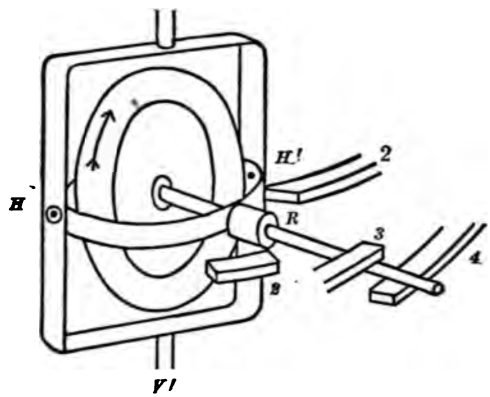

FIG. 16.

this position are in the same line, one wheel being placed on each side of the car at equal distances from the central line. The axles are held in a horizontal ring which can turn about a horizontal axis, HH', in a vertical frame, and this frame can turn about the vertical axis, VV', which is rigidly attached to the car. A roller, R, fits over a sleeve which is firmly attached to the horizontal ring, while the axle passes out through this sleeve, and extends some distance

outside. The curved contact surfaces, 1 and 2, which are firmly attached to the car, have their ends just over this roller, as shown in the figure. There is a slight space between the contact surfaces and the roller, so that normally the roller does not touch them, but a slight motion of the car, up or down relatively to the axle, will bring the roller into contact with one or the other of these surfaces.

If the gyroscope is spinning in the direction of the arrow, or in a negative direction, and the car starts to tip to the right, since the friction about the axles is slight, the wheel will not change its plane, while the car will at first move relatively to the gyroscope. But almost immediately this brings the contact surface 1 onto the roller, and as the tipping continues the axle will begin to be depressed. This velocity about the HH' axis will set up a gyroscopic couple about the vertical axis tending to turn the gyroscope in the direction R to H. The resulting velocity about the vertical will set up a gyroscopic couple about the horizontal axis tending to press the axle strongly against the contact surface. It will be seen that the depression of the axle gives rise to a precession about the vertical, and this natural precession opposes strongly any further tipping of the car. The case is precisely similar to that of the Obry device. The car in depressing the axle is opposed by the vertical precessional couple; in other words it has to overcome the heavy rotational inertia of the wheel. Referring back to Art. 7, and considering that, by the tipping, the gravitational component is hung upon the axle, the result will be that the axle precesses, but that it does not permanently fall, although it executes small nutations up and down. It maintains its level at the expense of the precession, but as long as the weight presses on the axle, it will continue to precess and thus would eventually become || to the length of the car. In such a position the car has no sta-

bility and would fall. Hence the simple arrangement of the two contact surfaces 1 and 2, with their natural precession, would not be sufficient to keep the car upright.

There are, however, two similar curved surfaces, which we shall call the guide surfaces, 3 and 4 in Fig. 16, which are placed with their ends above and below the extremity of the axle, and which are firmly attached to the car. These guide surfaces are at a slightly greater distance from the axle than the contact surfaces, so that when the car tips, the axle normally comes into contact with the contact surfaces, before it can touch the guide surfaces. But the ends of the contact surfaces extend only slightly beyond the middle of the axle in its normal position. When the axle precesses beyond the ends of the contact surfaces, however, the guide surfaces, 3 or 4, as the case may be, are brought on to it, and the precession continues. But this natural precession causes a mutual pressure between the axle and the guide surface, and as the axle is rotating, and in direct contact with the surface, which was not the case with the contact surfaces, frictional forces are developed which force the axle to *roll* along the surface. The natural precession now changes immediately to a forced artificial rolling precession, much greater than the former.

The natural precession along the contact surfaces was just sufficient to maintain the car at the level to which it had tipped, but this forced precession, being much greater, not only maintains the load, but actually forces it up rapidly to its original level of equilibrium (unstable). As soon as this position of equilibrium is reached, all pressure is removed from the axle, and there being no pressure between the axle and the surface, friction is abolished and the axle ceases to roll. If the momentum carries the car over to the other side, the axle is immediately engaged by the guide surface 4, or the contact surface 2, as the case

may be. The natural precession which causes the pressure between the axle and the surface it touches, which causes the rolling on the guide surfaces, which sets up the artificial precession, are now all reversed. When the car tips to the right, the contact surface 1, or the guide surface 3, comes into play: when the car tips to the left, the contact surface 2, or the guide surface 4, is engaged. There is thus a ceaseless precessional play about the vertical axis, now in one direction, now in another, with the position of equilibrium as a mean. The result is that the axle adjusts itself to a position of perpendicularity to the car, or to a position of maximum precessional efficiency, whenever the position of equilibrium is reached. The guide surfaces give the coarse, rapid, adjustment, while the contact surfaces give the finer adjustment.

The two flywheels, on either side of the car, rotate in opposite directions, or when viewed from the outside they rotate in the same direction. They thus naturally precess in opposite directions, but the frames are linked together so as to insure their being at the same angle with the length of the car at any instant. They are further geared together so that the two rotational velocities are precisely equal. The two flywheels work together in righting the car and the gyroscopic effect is thus doubled.

When the car rounds a curve, the link connection forces the frames to take the turn with the car, and no gyroscopic effect is produced by this turn of the car about a vertical axis, since the wheels rotate in opposite directions. The simple centrifugal force, however, throws the car over to the outward side of the curve. The gyroscope deals with this, as with any tipping. The appropriate guide surfaces are engaged and they bring the car to a position of equilibrium. But in this case the position of equilibrium is not the upright one, but where the centrifugal and gravita-

tional moments are equal and this position is towards the inner side of the curve. Hence the car automatically leans over to the inside, just as a bareback rider leans over towards the center of the ring. It automatically rights itself, immediately the curve is passed.

The car and the gyroscope are driven by electricity. With sufficiently great rotational energy, in proportion to the gravitational moment which it must overcome, the apparatus is perfectly stable.

Of course, there is an angle beyond which it would be impossible to raise the car, but with sufficient rotational energy and a not too great gravitational arm — a rather narrow car and weights kept well towards the middle line and low — the stability is assured. If the car were rounding a very sharp curve, and carried a great weight with its center of gravity high, it would incline to an extreme degree, and after rounding the curve might not be able to right itself again.

Such a car can travel over a taut wire cable as readily as an ordinary car does over a bridge. This gives it a marked advantage over the ordinary railroad in a rough, uncivilized country. It might even perform upon a slack wire, though it would be advisable to have all cable bridges firmly stayed. The reader will, of course, recognize that the work done in raising the car comes from the energy fund stored up in the flywheels, and that this as used up is replenished from the dynamo. Hence, in the end, it is the dynamo which raises the car. The axle rolls along its guide surface and pushes itself along, but it pushes itself against the weight of the car and thus must reduce its own rotation.

The gyro-car, while possessing a number of advantages over the two-rail car, is not without its element of danger. A storehouse of great energy is, like a powder magazine, always a source of potential danger to its immediate

vicinity. If the dynamo breaks down, the stored rotational energy would be sufficient to maintain the equilibrium for some time — perhaps for an hour. If the gyroscope breaks down, it is an end of everything. This can be provided against by using the best material and perfect workmanship. The cost of a gyro-car — each car necessarily carrying its own gyroscope and probably its own dynamo — will be much greater than that of the ordinary car, but this will be much more than offset by the greatly reduced cost of the roadbed. In its own special field the gyro-car possesses advantages over the ordinary car, which will eventually, though slowly, assure its adoption on a larger scale. At present the only working line in the world is in South Africa, though a number of other lines are projected.

23. The Anschuetz-Kaempfe Gyro-Compass.

The Anschuetz-Kaempfe gyro-compass is shown sectionally and schematically in Fig. 17. A housing, G, encloses the gyroscope, which rotates about a horizontal axle, shown by the projections from the housing. A hollow vertical stem is rigidly attached to this housing, and carries above, the compass card C, and just below the card, a flaring cone which ends in its lower border in a hollow circular chamber A, A. This hollow chamber is immersed in mercury contained in the hollow casing P, P. This hollow casing is hung on gimbals, two of the pivots of which are shown at P, P. The mercury is indicated by the dotted space. The hollow chamber, A, A, floating in the mercury, thus supports the weight of the gyroscope, with its housing, stem and compass card. The stem passes closely through the upper openings in the casing P, P, which are only a little larger than the stem, and thus keep the stem centered, but yet permit it to turn freely. The disc is an induction rotor of

the squirrel-cage type, and is driven by a tri-phase current. The wiring is shown schematically.

The diameter of the rotating disc is about 15 cm. (6 inches), and the velocity is 20,000 turns per minute. The

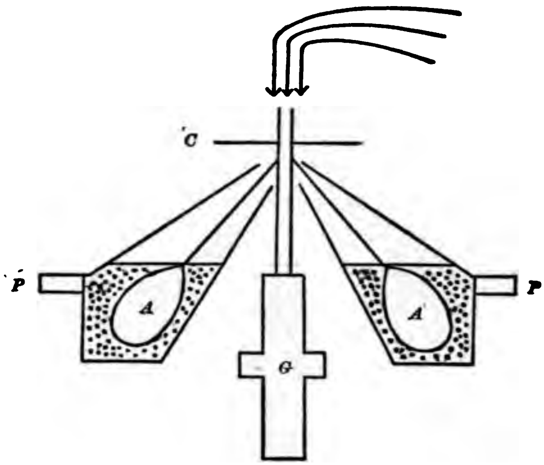

FIG. 17.

center of gravity is slightly below that of the displaced mercury, so that gravitationally it is stable. An arrangement for damping the swings about the vertical axis (not shown in the Fig.) is provided.

The case here is similar to that discussed in Art. 8. We can resolve the earth's rotation into that about a horizontal axis in the south-north direction, $\dot{\psi} \cos \lambda$, and that about a vertical axis, $\dot{\psi} \sin \lambda$, where λ is the latitude, and $\dot{\psi}$ the angular velocity of the earth. The rotation about the south-north axis, $\dot{\psi} \cos \lambda$, will urge the axis of the gyroscope into coincidence with itself, and this will eventually be accomplished through a series of swings on each side of the south-north line, just as a pendulum or magnetic needle swings on each side of its position of equilibrium, until it is brought to rest through friction. The directive couple

is, when the axis of the gyroscope is in an east-west direction, $C\omega\,\dot{\psi}\cos\lambda$, where $C\omega$ is its moment of momentum about its axis. The problem has already been worked out (Art. 8), and gives, for a rotational velocity of 20,000 per minute, under frictionless conditions and with the outer ring rigidly attached to the earth, about 11 seconds for the period at the equator.

In the present case the viscosity of the mercury and the skin friction of the immersed cone give rise to a very considerable resisting couple. Furthermore the moment of inertia, A, about the vertical axis, is not now simply that of the disc, but that of the whole floating mass, gyroscope and motor, housing, stem, card and floats, together with whatever mass of mercury it carries along with itself. The mass of the disc alone in the Anschuetz compass weighs 1500 grams (3 pounds).

The instrument is, in a certain sense, a gyroscopic pendulum, and the rotation about the vertical axis must be accompanied by small nutations in a vertical plane through the axis of the disc. Every fixed body on the earth's surface turns about a vertical axis with angular velocity $\dot{\psi}\sin\lambda$. The mercury, while not a fixed body, practically acquires this velocity from the sides of the containing case which is fixed, but a certain part of it is carried by the supporting chambers along with its rotation, and this, by its momentum, carries the rotation beyond the point where the gyroscope would have stopped. On the return swing, the gyroscope has to overcome this momentum and then start up a momentum in the opposite direction with like results on the other side. Hence damping arrangements are necessary.

In Art. 8 we saw that the period in the hypothetical frictionless case was $T = 2\pi\sqrt{\dfrac{A}{C\omega\dot{\psi}}}$, at the equator, or

about 11 seconds for a velocity of 20,000 turns per minute. In the present case the value of A becomes very much greater, and the directive couple is very much weakened by the frictional and inertianal drag.* Consequently the period must be very much greater, and is, in fact, for the Anschuetz compass about 70 minutes. Starting from rest, the instrument must be spun for three hours before it gives an indication. Placing a delicate spirit level on the card, the minute nutations are barely discernible, and this alone gives an indication to the eye that the card is moving. After once settling down to the line of the meridian, it keeps its position very well, and the oscillations are very small. It has been run several weeks at a time in trials on warships and has kept most of the time within $\frac{1}{2}^\circ$ of the true north. It is the first gyro-compass, among many previous attempts, that has proved successful. Even here the success has not been complete and it is rather to be looked upon as a pioneer than as the final word. The ideal compass would be one that could be used quickly when wanted, and then put out of commission; in other words a "stiff" compass with a short period. The present high cost of the Anschuetz compass — several thousand dollars per instrument — is prohibitive for merchantmen and for most naval ships. They will probably be produced at less cost in the future, but, as in the case of all inventions, there will be no supersession of the magnetic compass. Both will be used in their own respective fields. The modern dreadnought, which is a mass of steel, will have need of the gyro-compass to avoid its own excessive disturbing action on the magnetic compass.

This disturbing action in iron merchantmen can be avoided by placing the compass aloft on a pole mast. A compass 30 or 40 feet distant from any iron is practically

* Alcohol would be preferable to mercury, other things being equal.

unaffected. The directive force on a magnet is the product of its intensity of magnetization into that of the earth's horizontal field. Although the latter is weak, by increasing the former, a strong directive force is obtained. An electro-magnet used as a needle is practically dead beat. It springs into the magnetic meridian as soon as the current is turned on and stays there. A hollow electro-magnetic needle, immersed in alcohol and displacing a weight of the fluid equal to its own, has little friction on its axis and is unaffected by outside buffetings. This principle is used in nature in protecting the brain and spinal cord, the inner ear and the fœtus from outside shocks. Such a compass placed aloft could withstand the heavy motion it would experience there. A fore-and-aft coil, through which a current could be sent, would convert it into a tangent galvanometer which could be read from below. The method would be to measure the current necessary to deflect the needle first on one side, then on the other of its position — the true magnetic north — in order to have it make electrical contact with two light springs. The determination of a deflection angle, due to a known current passing through a galvanometer, is very accurate.

24. Ballistics.

A projectile fired from a rifled gun acquires a high rotational velocity about its long axis. For the largest shells this is over 100 turns per second, and for small arms 3000 per second and over. In following their trajectories, they always keep practically end on. This is the uniform result of observation, with the exception that in high angle firing — mortar firing — above a certain angle, the velocity becomes so small at the apex of the trajectory, and the curve so sharp, the resistance of the air is insufficient to turn the projectile, and it "tumbles."

A peculiar phenomenon of projectiles is known as the "drift." If the twist is to the right, the projectile drifts away from the vertical plane of fire to the right. If the twist is to the left, it drifts to the left of the plane of fire. The drift increases with the range and time of flight and is roughly proportional to the time of flight. It is thus possible to hit an object around a corner, which is concealed from the gunner.

In aiming it is necessary to allow for this drift, and sights are made so that the gun is pointed in an opposite direction by an amount increasing with the range. Observation shows that the axis of the projectile does not turn laterally following the lateral curve in the trajectory, but remains || to the vertical plane of fire.

For large shot the amount of the drift averages as follows:

Range	500 meters,	1000 meters,	2000 meters,	3000 meters.
Drift	.25 meters,	1.1 meters,	4.4 meters,	11.5 meters.

The facts then are as follows:

The axis of a projectile turns in its flight about a horizontal axis, thus following closely the vertical curve of the trajectory. It drifts away from the vertical plane of fire, but its axis remains practically || to this plane. It does not turn laterally appreciably, and it always strikes end on. The drift has never been explained. Several attempts have been made to explain it as a gyroscopic phenomenon, but these have all failed.* One explanation is that as the axis begins to make an angle with the trajectory, the air resistance lifts its nose a little and this results in a gyroscopic couple which turns it laterally. Why it remains thus laterally "slewed," and the air resistance does not form another couple making it turn about a horizontal axis, is not stated. It is added

* Klein and Sommerfeld, "Ueber die Theorie des Kreisels," Crabtree, "Spinning Tops and Gyroscopic Motion."

that the reason the nose is lifted is because all objects moving through a resisting medium tend to present their broad surfaces to the line of motion, and that the only reason a projectile does not turn broadside on is because of the rifling. And then we have the universal idea, both among mathematicians and the laity, that the purpose of rifling is "to keep the projectile end on and to steady it in its flight."

It is true that thin flat objects tend to present their broad surfaces to the line of motion. A coin dropped in water always falls flat on the bottom. A card thrown into the air always turns in falling so as to bring its surface $\perp$ to the line of fall. It balances for an instant in this position, when it slides off again in another direction, but brings up by presenting its surface again, and thus by a series of swings, between which it comes nearly to rest, it gradually falls to the ground.

But the facts are directly opposite with a long fusiform body. Here the resistance of the air in proportion to the velocity forms a strong couple forcing the long axis into coincidence with the trajectory. A javelin or spear, hurled forcibly, corrects any initial slew, and following the trajectory, strikes end on. An arrow, even untipped, keeps end on. Bolts were fired from guns long before rifling was thought of, and there was never any difficulty in keeping them end on. A long cylindrical pointed bullet can be fired from a smooth bore with the same result. If there is the least slewing, the air is banked up on the forward side and on the rear side a high vacuum is formed. Thus a powerful couple tends to turn it straight again. In fact, it is easier for a non-rotating projectile to keep end on, with sufficient velocity, than for a rifled one, and it follows its trajectory closer than does a rifled projectile.

What then is the object of rifling, if it is not to keep the

projectile end on? That is a matter of interior ballistics. By delaying the time of the projectile's leaving the gun; by letting the gases expend a part of their energy in communicating a rotation to the projectile, instead of driving it off at one puff out of their action, the slow-burning powder has time to develop its full pressure, and when the projectile finally leaves the gun, it has a much higher velocity than it otherwise could have had. Further, it possesses the added energy of rotation, which adds to its shattering effect on any resisting object. When rifling was first introduced, it was noted that the range was enormously increased, but it was not noted that projectiles kept end on any better. It is true that the higher velocity, other things being equal, tends to this result, but not the rifling *per se*.

The first effect of the air resistance, when the axis deviates from the trajectory, is to turn the point *down*, not up as in the previous explanation. Referring to Art. 9, we see that the axis executes a looped curve in obedience to the ⊥ couple, and at first does not deviate far from the plane of the couple. In the case of small arms, where the trajectory is flat and the turning couple small, we should have something like that represented in Fig. 8. The axis would remain close to the vertical firing plane while executing a series of loops. Where the trajectory is higher and the turning couple greater with the rotational velocity less, as with heavy shot, we should have the case represented in Fig. 18. The axis would work away from the vertical plane more rapidly, but would be instantly met by a lateral couple. In fact, the axis of the turning couple, always nearly ⊥ to the axis of the projectile, would revolve around this axis very rapidly, meeting it on all sides promptly as it deviated from the line of flight, causing it to execute a series of major loops, each loop corresponding to a complete

revolution of the air couple around the trajectory. On these major loops would be superposed a series of minor loops, corresponding to the minute deviations from the average path of the point. We have already discussed these loops in Art. 9.

In this manner, the predominant couple, viz., the couple downward, would force the axis downward, but the actual couple at any instant would always be turning about the axis and forcing it to execute the series of major loops tangent to the vertical plane. These loops, both the major and minor loops, are executed very rapidly and are very small. The proportions in Fig. 18 are naturally much exaggerated. Consequently, the axis will remain nearly in the vertical firing plane. The drift, then, cannot be explained as a gyroscopic phenomenon.

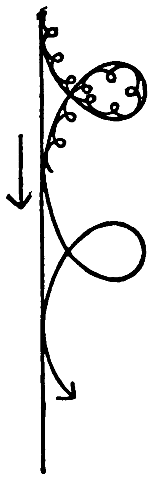

FIG. 18.

The major loops resemble a trochoid as do the minor loops, though the loops in the latter are not all exactly equal. We might call the curve described by the axis as a trochoid of the second order, or a trochoid within a trochoid. The result is that a rifled projectile "wobbles" in a very narrow cone about the line of flight, while an unrifled projectile does not wobble.

When the axis inclines to the trajectory and the powerful air couple is pushing the 'butt' end *up*, the air is heavily compressed at the 'butt' and on the lower side, while on the upper side it is nearly a vacuum. Hence, by its own rotation, the projectile *rolls* sideways on a heavy cushion of air beneath it, while it suffers little or no frictional resistance on its upper side. The frictional resistance on the under side is proportional to the square of the rotational

velocity and is considerable. Hence the projectile "drifts" or *rolls* to the right or left according to its rotation, while the axis always remains practically || to the firing plane. The case is somewhat similar to that of a single propeller already mentioned, where the resistance on the lower blade being greater than that on the upper, a side thrust results.

25. Astronomical Applications.

We have seen that on the supposition that the earth is a perfectly rigid body, the average inclination of its axis to the ecliptic cannot change. It undergoes forced nutations, due to the revolutions of the sun and moon and to the precession of the moon's orbit, but these nutations are periodically recurring quantities and cannot permanently affect the inclination of the axis.

Now geologists have had to face the stubborn fact that there have been alternating periods of warmth and cold on the earth. At one time a tropical fauna and flora has flourished in the vicinity of the poles; at another time the earth has been glaciated for considerable distances from each pole. Hence innumerable "theories" have been formed to explain these changes. They all start from the hypothesis that the earth being a rigid body, its inclination to the ecliptic can never change. They then proceed to manufacture some miraculous machinery by which a tropical flora can be made to flourish for six months around the pole without any light, i.e., through the long arctic night. They are careful to supply plenty of heat to their tropical greenhouse, as if that were all-sufficient and plants could grow without actinic rays.

Perhaps the simplest of all is that of Poisson. According to Poisson, the solar system at times passes through warm regions in space, and at other times it passes through cold

regions of space: which is very nearly the same as saying that the glacial and genial periods occurred because it was colder and warmer at the respective times. The most recent "theory," which has appeared within the last few months, is that the glacial epochs were due to volcanic dust in the atmosphere. During periods of great volcanic activity, the dust in the atmosphere is supposed to reflect away the greater part of the sun's heat with a resulting glaciation. To be sure this does not explain the tropical flora that once flourished near the pole, but what does that matter, if it "*explains*" the glaciation? There is no room here, and it would be utterly unprofitable to discuss or even tabulate the various "theories" advanced.

The fact of the occurrence of a tropical flora in the vicinity of the poles, of which there is no doubt, proves conclusively that at that time the axis of the earth was $\perp$ to the ecliptic. There is no escaping it. No "Greenhouse Theory," no "Variation in the Eccentricity of the Earth's Orbit," no "Zonal Belt Hypothesis" or other machinery can account for it. There is no doubt that the pole of the earth oscillates through the pole of the ecliptic. If the result of a mathematical analysis shows that the earth's axis cannot change its inclination, as we have proved in the first part of this book, then, when confronted with the fact that it *does* change its inclination, we can only say that the assumptions in our problem do not agree with the actual case. We have postulated in all our previous work that the gyroscope must be perfectly rigid. Now there are no perfectly rigid bodies in nature. All bodies are more or less elastic, the earth among them.

The problem of an elastic rotating body under the attraction of another distant body, has never been worked out. Its complete solution would be rather complicated

and we shall not attempt it here. We may, however, point out certain probable results. We have seen (Art. 4) that the compound centrifugal force, due to a combined rotation and revolution, produces a solid tide in an elastic body which is always in the plane of the orbit and directed towards the attracting body.* Now the result of such a tide is that there is a moment about an axis ⊥ to the orbit equal to the mass of the tide, into the square of its distance from the center of the earth into the orbital velocity. In other words, there is a very minute rotation about an axis ⊥ to the ecliptic and in a positive direction. The resulting motion would, therefore, be something like that of a perfectly rigid body which not only rotated about its axis, but also had a minute rotation about an axis ⊥ to the orbital plane. This case has been treated in Art. 8. Or it is remotely similar to the case of the top, already mentioned, which rolls on its "toe," and thus sets up a moment about a vertical axis, thereby causing the top to rise.

Such a motion cannot be stable. The axis of figure will strive to set itself into coincidence with the axis ⊥ to the ecliptic, and will, in fact, do so. But its poleward velocity, just as in the gyro-compass, will carry it beyond, and thus it will oscillate backwards and forwards across the pole of the ecliptic, but by lesser and lesser distances. The solid tide will eventually reduce the rotation period to that of the revolution. When the elastic body, therefore, has arrived at a position of permanent perpendicularity to the ecliptic, and its rotational period is equal to its revolutional period, it will have attained permanent stability of motion, and not before.

The earth, according to estimates which cannot be

* The effect of the tides of the shallow mobile oceans, which do not keep phase with the attracting body, is insignificant in comparison with the solid tide.

considered very exact, is now decreasing its obliquity by something like half a second a year. Whatever it is, it is certainly very small and only measurable over a number of years. According to Ptolemy the inclination was $24°$, 2000 years ago, and perhaps this is near the truth. It is now $23° 27' 08''$.

It must be remembered that the plane of each planetary orbit precesses under the action of all the others. The resulting motion is slight, but extremely complicated and without regular period. The observed decreasing obliquity of the earth's axis has been explained as due to this changing inclination of the ecliptic — in other words to its precession. Lagrange even calculated that the obliquity would decrease for 15,000 years, reaching a value of $22\frac{1}{4}°$, after which it would increase, but that the total variation could never exceed $1.2°$ on each side of a mean. Leverrier found much narrower limits. Some of the assumptions of Lagrange are no longer tenable.

The earth undoubtedly possesses a high degree of rigidity, but it is not perfectly rigid. There can be no reasonable doubt that it oscillates through the pole of the ecliptic, and the period need not be very enormous, as astronomical æons go, although it is probably many thousands of years. If we take the present supposed rate and go back only 40,000 years, we find that the obliquity would have been some $5°$ greater than now—enough to cause a glacial epoch.

26. Geological Applications.

Geologists used to believe, and some do yet, that the inequalities of the earth's surface are due to the attempted adaptation of the crust to a shrinking core, from cooling. Mathematicians have shown, and it is not hard to do so, that such an effect would be entirely inadequate to produce

the observed results. The favorite comparison is to a wrinkled apple.

The main ridges and depressions on the earth's surface are approximately along meridians and east-west lines, and their diagonals— a fact which would lead a mind with what we might call a "dynamical instinct" to suspect that they were connected with the rotation of the earth, i.e., to catastrophic changes in the rate of rotation. If the rotation of the earth were suddenly diminished (or increased) by even a very small amount, the result would be a throwing up of mountain chains and hollows in the general directions which are now observed. These disturbances would, of course, be a maximum at the surface and less and less as the center was approached. With this violent upheaval, there would be tremendous fracturing, much production of heat, liquefaction and volcanic activity extending to a considerable depth into what is called the crust. The amount of rotational energy lost would in fact be equal to the work done, plus the heat.

Geologists have clung tenaciously to a belief in a molten fluid interior of the earth. Lord Kelvin and Prof. Darwin, on the other hand, from an examination of the ocean tides, have found that the longer as well as the shorter periods of the tidal fluctuations appear plainly, and arguing that the longer periods should disappear if the interior of the earth is viscous, have concluded that the rigidity of the earth must be greater than that of glass, though less than that of steel. While their work and reasoning is not wholly convincing to the author, there seems to be a possibility that both views may be correct. We do not know what the properties of matter may be, under the enormous pressures and temperatures which probably exist in the earth's interior. Our experience of matter is solely as it exists at the earth's surface. Whether a critical tempera-

ture exists above which gases cannot be liquified even by pressures far beyond those of our laboratories, we do not know; nor do we know whether bodies under conditions extreme from the state in which we know them, may be plastic and yet possess qualities like our rigid bodies.

If we suppose the interior of the earth plastic, then, supposing that this interior were divided into similar spheroidal shells, it would result that the precessional motion could not be equally shared by all the shells. There would be a slight movement from time to time of certain parts over other parts. This dissimilarity of motion could not long continue from the fact that the precessional rotation is strictly about the $\dot{\psi} \sin \theta$ axis and this could not proceed far in the case of one spheroidal shell moving over another without distorting it.

Such inequalities of motion would produce gyroscopic forces which would lead to sudden, violent, catastrophic readjustments, resulting in the conversion of rotational energy into heat and distortional work. In the early ages of the world its interior was probably more plastic than at present, and these distortions were probably more violent, resulting generally in the conformations at present existing. That the process is still at work, we can judge from certain recent world-wide readjustments. It is, therefore, suggested that the past distortions of the earth's surface had their origin in gyroscopic action on a plastic interior, due to a lack of uniformity of precession of the different layers of this interior — in other words to interior gyroscopic action, making itself felt chiefly in the outer layers. What are simply internal strains in a rigid gyroscope become motion in a plastic gyroscope.

27. Meteorological Applications.

Cyclones are rotating masses of air, which always rotate in the same direction as the surface of the earth under them. They have, at any time, as definite a rotational moment as if they were solid. They are, therefore, gyroscopes, for we have defined a gyroscope simply as a rotating mass, and a gyroscope may be solid, liquid or gaseous. Fluids in motion acquire some of the properties of a solid, among them that of shape or form. Indeed, according to advanced modern ideas, all matter may consist of discrete particles which by their motions about each other, impart to their congeries the different properties which we recognize in them. A flimsy piece of paper which cannot be considered to have any stable form, if rotated rapidly about an axis through its center, becomes practically a rigid body. It possesses rigidity and elasticity. It is not easily bent, and when bent springs back again to its original shape. If struck a sharp blow it resonates like a rigid body, giving out a clear sound.

A cyclone has a very definite form, which it preserves for an appreciable time, and it is that of an oval disc. It moves from place to place and is as much an entity as if it were a solid body. The greater its rotational velocity the more nearly does it approach dynamically a rotating solid, and we can apply, to a sufficient degree of approximation, the same formulas we have used for solid gyroscopes. And we shall do this, although a prominent meteorologist once wrote the author that "to consider a cyclone a gyroscope, is a misconception of nature."

Since at the same time that a cyclone is rotating about its axis, it is being carried around the axis of the earth, it is evident that gyroscopic forces will be set up. Since in the northern hemisphere both these motions are positive, and

in the southern hemisphere both are negative, it is evident that in both cases, at first at least and generally, cyclones move towards their respective poles. Heavy tropical hurricanes, or cyclones, always move at first to the north in the northern hemisphere, and to the south in the southern hemisphere. They perform this motion because of gyroscopic action: in other words, because they are gyroscopes.

By an analysis of the motions of actual cyclones, it is found that they maintain, during the greater part of their run, their moment of momentum about their axes tolerably constant. This, of course, applies chiefly to those with a great amount of rotational energy — heavy tropical cyclones — and we shall confine ourselves to a consideration of these alone. A top with a small moment of momentum is easily disturbed by outside forces and its energy quickly dissipated by friction, so that, in such a case, it would be impossible to predict its motion. The same applies to cyclones.

If we have a complete knowledge of the forces at work on a cyclone, we can predict its motion (or path) as readily as that of any other gyroscope. The chief outside forces are frictional forces between its under surface and the earth, and, as in the case of all frictional forces, the formulas are more or less empirical. In working out mathematical problems we usually postulate certain ideal conditions. There is no friction: bodies are perfectly rigid or perfectly elastic, etc., while in applying these formulas to actual cases, we are met by forces and conditions that often are not perfectly measurable and rather variable. The application of a mathematical theory to practice, can, therefore, rarely give exact results, but as approximations they are useful according to their nearness to what is observed. On the other hand, the comparison of what is observed with what should have been the result of rigid theory, although

there can never be exact agreement, still is a criterion by which reversely to judge the theory. When Newton found that by using the at-that-time accepted distance of the moon, her motion could not be reconciled with his theory, he decided that the distance, as assumed, must be correct, and immediately gave up his theory. Later it was found that the distance assumed was wrong, while the theory was correct.

By applying the gyroscopic theory to the motion of a cyclone, we find the agreement between theory and observation to be what we might expect if the theory is true. It is thus possible, by using assumptions as to friction which may be only approximately correct, to predict the motions of these gyroscopes with considerable accuracy, and certainly within the limits of usefulness. There is not space here to discuss fully these motions, which relatively to the earth — that is, the path traced out by their axes on the earth — are extremely regular smooth curves, generally paraboloid in form, with their axes horizontal. A fuller discussion of their formation, characteristics, and motion can be found in the author's book on "The Atmosphere."

It is only yesterday that certain practical applications of the gyroscope have astonished a wondering and mystified public. Man has taken the child's toy and made it carry a train of cars over a tight rope, steadied a ship so that its passengers no longer suffer from seasickness, found the true north when masses of iron rendered a magnetic compass useless, and sent a torpedo straight to its mark. These are probably but beginnings. It is to be hoped that among the future applications of the gyroscopic principle, one, which will certainly not be the least useful, will be the prediction of cyclone paths. A cyclone is observed in the West Indies. Quickly its future course is published. It will be at this point on the next day; it will be at that point one week after. It is possible to do this approximately, and such a valuable means of weather prediction should not be ignored.

NOTE.

TRI-AXIAL BODIES UNDER EXTERNAL FORCES.

In Art. (7) we saw that a bi-axial body with high rotational velocity described a small circle about a point, while this point moved with a constant precessional velocity about the vertical, and at a constant inclination. The angular velocity of the axis in the circle was $\frac{C\omega}{A}$, and the constant precession of the center was $\frac{mgl}{C\omega}$. This was on the supposition that the motion was in a plane surface. Actually, the surface is part of a sphere and the angular velocity in the circle is $\frac{C\omega}{A \cos \gamma}$, where γ is the angle subtended by the radius, and $\cos \gamma$ differs from unity by the same order of small quantities as those we have previously neglected. This relation shows that the body executes a Poinsot motion about the center as an invariable line.

In the same manner, for a tri-axial body, we may consider the motion to be a Poinsot motion about a point, while this point moves with a constant precessional velocity about the vertical. Let G be the constant moment of momentum about the Poinsot axis, $\dot{\psi}$ the constant precession of this axis, and θ its inclination. Then, $\dot{\psi} \sin \theta G = mgl \sin \theta$, or the gyroscopic couple due to this moment of momentum, combined with the constant precession, obliterates the gravitational couple.

The Poinsot motion is limited by the two precessional circles made by the two bi-axial bodies having moments of inertia C and B, and C and A. These circles have radii

$$\frac{mgl \sin \theta B}{C^2\omega^2} \quad \text{and} \quad \frac{mgl \sin \theta A}{C^2\omega^2}\text{, respectively.}$$

NOTE.

ON THE MOTION OF CYCLONES.

The earth being a rotating spheroid, every particle on its surface is subjected to a tangential gravitational component towards the pole, and to a tangential centrifugal component towards the equator. The amount of the latter is $R \sin\theta \cos\theta \dot{\psi}^2$, where R is the radius of the earth, and $\dot{\psi}$ the horizontal angular velocity of the particle at any instant. Hence a movable particle, at rest relatively to the earth, must have a gravitational component towards the pole equal to $R \sin\theta \cos\theta \omega^2$, where ω is the angular velocity of the earth. When the horizontal angular velocity of a particle is greater, or less, than that of the earth, the particle will experience an acceleration along a meridian, towards the equator or towards the pole, as the case may be, and the amount of this acceleration is $R \sin\theta \cos\theta\,(\dot{\psi}^2 - \omega^2)$.

In the case of a cyclone, every particle in it will experience this meridianal acceleration according as the value of $\dot{\psi}$, which it possesses at any instant, is greater, or less, than ω. The sum of these accelerations will be the total meridianal force acting upon the cyclone.

Considering, at first, only a ring of rotating matter, let $\dot{\psi}_c$ be the horizontal angular velocity of its center, r the radius of the ring, ϕ the angle which a particle makes at any instant with a parallel of latitude through the center, and $\dot{\phi}$ the angular velocity of the ring relatively to this parallel. Then the horizontal angular velocity of a particle is

$$\dot{\psi} = \dot{\psi}_c - \frac{r\dot{\phi}\sin\phi}{R\cos\left(\theta_c + \dfrac{r\sin\phi}{R}\right)},$$

where θ_c is the latitude of the center,

$$\dot{\psi}^2 = \dot{\psi}_c^2 - \frac{2\dot{\psi}_c r\dot{\phi}\sin\phi}{R\cos\left(\theta_c + \dfrac{r\sin\phi}{R}\right)} + \frac{r^2\dot{\phi}^2\sin^2\phi}{R^2\cos^2\left(\theta_c + \dfrac{r\sin\phi}{R}\right)}.$$

Hence

$$R\ddot{\theta} = \frac{R}{2}\sin 2\left(\theta_c + \frac{r\sin\phi}{R}\right)(\omega^2 - \dot{\psi}_c^2) + 2\dot{\psi}_c r\dot{\phi}\sin\phi\sin\left(\theta_c + \frac{r\sin\phi}{R}\right)$$
$$- \frac{r^2\dot{\phi}^2\sin^2\phi}{R}\tan\left(\theta_c + \frac{r\sin\phi}{R}\right).$$

Since $\dfrac{r\sin\phi}{R}$ is, in a cyclone, usually a small quantity, in developing a

function of this angle by Taylor's theorem, we can neglect its squares and higher powers without appreciable error.

Hence

$$R\ddot{\theta} = \frac{R}{2}\left[\sin 2\theta_c + 4\cos 2\theta_c \frac{r\sin\phi}{R}\right](\omega^2 - \dot{\psi}_c^2)$$
$$+ 2\dot{\psi}_c r\dot{\phi}\sin\phi\left(\sin\theta_c + \frac{r\sin\phi}{R}\cos\theta_c\right)$$
$$- \frac{r^2\dot{\phi}^2\sin^2\phi}{R}\left(\tan\theta_c + \frac{r\sin\phi}{R}\sec^2\theta_c\right).$$

The sum, or integral, of all these accelerations, for the complete ring, is

$$\pi r R\sin 2\theta_c(\omega^2 - \dot{\psi}_c^2) + \frac{2r^3}{R}\cos\theta_c \pi\dot{\psi}_c\dot{\phi} - \frac{r^3\dot{\phi}^2}{R}\pi\tan\theta_c = MR\ddot{\theta}_c.$$

Putting A for the moment of inertia of the ring about the center of the earth, since $2\pi r$ is the mass of the ring, we have

$$A\sin\theta_c\cos\theta_c(\omega^2 - \dot{\psi}_c^2) + C\dot{\phi}\dot{\psi}_c\cos\theta_c - \frac{C\dot{\phi}^2}{2}\tan\theta_c = A\ddot{\theta}_c. \qquad (1)$$

We, therefore, see that the meridianal forces acting upon a cyclone are equivalent to four forces acting on its center, *viz.*, a gravitational force acting towards the pole, a *revolutional* centrifugal force acting towards the equator, a gyroscopic force ($C\dot{\phi}\dot{\psi}_c \cos\theta_c$) acting towards the pole, and a *rotational* centrifugal force acting towards the equator.

If it should happen that these four forces (two positive and two negative) were balanced, or $\ddot{\theta} = 0$, then the cyclone would remain on the same parallel of latitude. This rarely happens, for, when $\dot{\phi}$ is large, the gyroscopic force, at first, always urges the cyclone towards the pole. Again, we see from Equation (1) that it cannot move continually towards the pole, but must reach a latitude—its latitude of equilibrium—where $\ddot{\theta}_c$ vanishes. Here it performs a quasi Poinsot motion about the axis of the earth, providing there are no frictional forces.

For the hypothetical frictionless case, our equations of motion are,

$$A\sin\theta\cos\theta(\omega^2 - \dot{\psi}^2) + C\dot{\phi}\dot{\psi}\cos\theta - \frac{C\dot{\phi}^2}{2}\tan\theta = A\ddot{\theta}. \qquad (2)$$

$$A\dot{\psi}\sin\theta\dot{\theta} - C\dot{\phi}\dot{\theta} = AD_t(\dot{\psi}\cos\theta). \qquad (3)$$

Integrating these equations, we have,

$$A\left(\frac{\sin^2\theta}{2} - \frac{\sin^2\theta_0}{2}\right)\omega^2 + \frac{C\dot{\phi}^2}{2}\log\frac{\cos\theta}{\cos\theta_0} = \frac{A\dot{\theta}^2}{2} + \frac{A\dot{\psi}^2\cos^2\theta}{2} = T \qquad (4)$$

and

$$C\dot{\phi}\sin\theta + A\dot{\psi}\cos^2\theta = C\dot{\phi}\sin\theta_0 + A\dot{\psi}_0\cos^2\theta_0. \qquad (5)$$

Equation (4) is the energy equation, and Equation (5) expresses the fact that the moment of momentum about the earth's axis remains constant, since the forces are wholly meridional.

In dealing with actual cyclones, we find that their motions are greatly influenced by rotational friction. The chief factors in the friction of gases moving over liquid or solid surfaces, are the relative velocity of the opposing surfaces and the amount, or area, of such surfaces engaged. It is probable, that the friction is proportional to the square of the relative velocity between the two surfaces, so that the friction due to the relatively small translational velocity of the cyclone as a whole, may be neglected in comparison with that due to the high rotational velocity. Hence, if the meridianal forces are balanced, a cyclone may slide along a parallel of latitude—east or west, according to its original velocity—without much diminution of its horizontal velocity relatively to the earth, although, of course, there is some retardation.

If the cyclone is circular, and its stream lines are symmetrically arranged about the center, then the rotational friction can have no effect in moving it : it is merely a retarding couple about the axis of the cyclone which is continually overcome by fresh accessions of energy from precipitation. In this manner a heavy cyclone maintains its rotational energy practically constant during most of its run. When, however, the gyroscopic force urges the cyclone towards the pole, since this force acts chiefly at the center where the rotational energy is greatest, the axis is pulled away from its central position towards the advancing edge, and the cyclone becomes deformed into an oval. The rotational friction now comes into play. Since the equatorial half has a much greater area than the polar half, the preponderating friction on the equatorial side thrusts the cyclone towards the west, and decreases markedly its horizontal velocity.

There is a sliding rolling, as it were, along a parallel of latitude. Hence we find that as long as a cyclone remains upon the same parallel, its horizontal velocity remains nearly constant, and the cyclone is practically circular in form with its axis in the center. When it moves away from a parallel of latitude, its horizontal velocity begins to change, decreasing as it moves towards the pole, and increasing as it moves towards the equator. The latter motion, as we have already pointed out, is very rare. There are no exceptions to this rule.

Not knowing the law of frictional retardation, we must have recourse to empirical formulae. The formula $v_h = a - b \tan \theta$, where v_h is the horizontal velocity, and a and b are constants to be determined from observation at two positions, is applicable to most cyclones.

Let us apply this formula to the Porto Rican Hurricane of August, 1899—one of the most severe cyclones of recent years. See cut on p. 104.